Turbofan Sizing and Analysis with Variable $c_p(T)$

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1 Summary

The turbofan model described here is used for two purposes:
1) Sizing of a turbofan engine to obtain a specified thrust at design conditions, and
2) Calculations for a given engine at off-design conditions, with a specified thrust or $T_{t\,4}$.

It is largely based on the formulation of Kerrebrock [1], with a number of modifications. Turbine cooling flow which bypasses the combustor is introduced, and a multi-constituent gas model with variable $c_{p(T)}$ is used for all the flowpath calculations.

2 Nomenclature

Variables

A	flowpath area
a	speed of sound ($= \sqrt{T R\, c_p/(c_p - R)}$)
f_f	fuel mass flow fraction ($= \dot{m}_\mathrm{fuel}/\dot{m}_\mathrm{core}$)
f_o	mass flow offtake fraction ($= \dot{m}_\mathrm{offt}/\dot{m}_\mathrm{core}$)
f_c	total turbine-cooling mass flow fraction ($= \dot{m}_\mathrm{cool}/\dot{m}_\mathrm{core}$)
F	net effective thrust force
M	Mach number ($= u/a$)
$\bar{m}$	component corrected mass flow ($= \dot{m}\,\sqrt{T_t/T_\mathrm{ref}}/(p_t/p_\mathrm{ref})$)
$\bar{N}$	component corrected rotation speed ($= N/\sqrt{T_t/T_\mathrm{ref}}$)
$\dot{m}_\mathrm{core}$	core mass flow through LPC
$\dot{m}_\mathrm{offt}$	bleed mass flow offtake, assumed to be at LPC discharge
P_offt	shaft power offtake from LPC or fan
P_prop	net effective propulsive power
h, h_t	static and total complete enthalpy
p, p_t	static and total pressure
T, T_t	static and total temperature
u	velocity
α	bypass ratio ($= \dot{m}_\mathrm{fan}/\dot{m}_\mathrm{core}$)
ε	blade row cooling flow fraction relative to compressor air
$\Delta h_{(\,)}$	total enthalpy jump across component ( )
$\pi_{(\,)}$	total pressure ratio across component ( )
$\eta_{\mathrm{pol}_{(\,)}}$	polytropic efficiency of component ( )
$\eta_{(\,)}$	overall total-to-total efficiency of component ( ), except for combustor
η_b	combustor efficiency, or fraction of fuel undergoing combustion
$\epsilon_\mathrm{l}, \epsilon_\mathrm{h}$	spool windage, bearing, and gear power loss fractions
$c_{p_i}(T)$	specific heat of gas constituent i
$h_i(T)$	complete enthalpy of gas constituent i
$\sigma_i(T)$	entropy-complement function of gas constituent i ($= \int (c_{p_i}/T)\,\mathrm{d}T$)
R_i	ideal-gas constant of constituent i
$\alpha_i, \beta_i, \lambda_i$	constituent i mass fractions for air, fuel vapor, combustion product

The constituent property values and the mass fractions will also be denoted as a vector, e.g.

$$\alpha_i = \vec{\alpha} \quad , \qquad h_i = \vec{h} \quad , \qquad c_{p_i} = \vec{c}_p \qquad \ldots$$

where i is the constituent gas index.

Subscripts

$(\ )_{\mathrm{f}}$	fan quantity
$(\ )_{\mathrm{lc}}$	low pressure compressor (LPC) quantity
$(\ )_{\mathrm{hc}}$	high pressure compressor (HPC) quantity
$(\ )_{\mathrm{ht}}$	high pressure turbine (HPT) quantity
$(\ )_{\mathrm{lt}}$	low pressure turbine (LPT) quantity
$(\ )_{\mathrm{fn}}$	fan nozzle quantity
$(\ )_{\mathrm{tn}}$	turbine nozzle quantity
$(\ )_{\ldots\mathrm{D}}$	design-case quantity

3 Pressure, Temperature, Enthalpy Calculations

3.1 Standard relations

The standard constant–c_p equations connecting a baseline state T_o, h_o, p_o to some other state T, h, p are the familiar caloric and isentropic relations, with the latter possibly having a polytropic efficiency η_{pol} included to account for a non-isentropic process.

$$\gamma \equiv c_p/(c_p - R) \tag{1}$$

$$\Delta h \equiv h - h_o = c_p\,(T - T_o) \tag{2}$$

$$\pi \equiv \frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{\eta_{\mathrm{pol}}{}^{\pm 1}\gamma/(\gamma-1)} \tag{3}$$

The $+1/-1$ exponent on η_{pol} indicates a compression/expansion process, respectively. These relations will be used for air close to ambient conditions, for which $c_p = 1004\,\mathrm{J/kg\text{-}K}$ and $\gamma = 1.4$ are nearly constant.

3.2 Gas mixture properties

For a gas or gas mixture with a temperature-dependent $c_{p(T)}$ relations (2)–(3) are not valid, and will be replaced here. The overall gas-mixture functions $c_{p(T)}$, $h(T)$, $\sigma(T)$, R are computed using the individual $c_{p_i}(T)$, $h_i(T)$, $\sigma_i(T)$, R_i constituent functions and the mass fractions α_i, β_i, λ_i. For air we have

$$c_p(T) = \sum_i \alpha_i\, c_{p_i}(T) = \vec{\alpha} \cdot \vec{c}_p(T) \tag{4}$$

$$h(T) = \sum_i \alpha_i\, h_i(T) = \vec{\alpha} \cdot \vec{h}(T) \tag{5}$$

$$\sigma(T) = \sum_i \alpha_i\, \sigma_i(T) = \vec{\alpha} \cdot \vec{\sigma}(T) \tag{6}$$

$$R = \sum_i \alpha_i\, R_i = \vec{\alpha} \cdot \vec{R} \tag{7}$$

while for fuel vapor β_i is used instead of α_i, and for the combustion products λ_i is used instead of α_i. The combustion relations and the calculation of λ_i are derived in the related document *Thermally-Perfect Gas Calculations*. Also derived are replacement relations for (2)–(3) for the variable–c_p gas. All these results will be summarized in the subsequent sections here. The function Jacobian derivatives ∂(output)/∂(input) will also be derived for each case. These are required for off-design performance calculations via the Newton method.

3.3 Enthalpy prescribed

Occasionally it is necessary to obtain the temperature from a specified enthalpy. This is performed by inverting the $h(T)$ function via the Newton method.

$$\text{initial guess:} \qquad T \;=\; T_{\text{guess}} \quad \text{(e.g. standard temperature)} \tag{8}$$

$$\text{solve:} \quad \mathcal{R}_T(T\,;\,h_{\text{spec}}) \;\equiv\; h(T) \;-\; h_{\text{spec}} \;=\; 0 \qquad \rightarrow \qquad T \tag{9}$$

The overall calculation will be denoted by

$$T \;=\; \mathcal{F}_T\left(\vec{\alpha}, h_{\text{spec}}\,;\,T_{\text{guess}}\right) \tag{10}$$

where the $\vec{\alpha}$ argument is required to evaluate the $h(T)$ function in (9), via (5).

The derivative of the calculated temperature is simply the inverse of the specific heat.

$$\frac{\mathrm{d}T}{\mathrm{d}h} \;=\; \frac{1}{c_p(T)} \tag{11}$$

3.4 Pressure ratio prescribed

For prescribed $p_o, T_o, \pi, \eta_{\text{pol}}$, the new state p, T, h after a compression/expansion process is computed as follows.

$$\sigma_o \;=\; \sigma(T_o) \tag{12}$$

$$c_{p_o} \;=\; c_p(T_o) \tag{13}$$

$$\text{initial guess:} \qquad T \;=\; T_o\,\pi^{R_o/(c_{p_o}\,\eta_{\text{pol}}{}^{\pm 1})} \tag{14}$$

$$\text{solve:} \quad \mathcal{R}_p(T\,;\,p_o, T_o, \pi) \;\equiv\; \frac{\sigma(T) - \sigma_o}{R} \;-\; \frac{\ln \pi}{\eta_{\text{pol}}{}^{\pm 1}} \;=\; 0 \qquad \rightarrow \qquad T \tag{15}$$

$$\text{then:} \qquad p \;=\; p_o\,\pi \tag{16}$$

$$h \;=\; h(T) \tag{17}$$

The solution for T is via Newton iteration. The overall calculation will be denoted by

$$\{p, T, h, c_p, R\} \;=\; \mathcal{F}_p\left(\vec{\alpha}, p_o, T_o, \pi, \eta_{\text{pol}}{}^{\pm 1}\right) \tag{18}$$

The function Jacobian derivatives $\partial\{p, T, h\}/\partial\{p_o, T_o, \pi\}$ are obtained by first implicitly differentiating the residual function (15) with respect to the specified p_o, T_o, π.

$$\frac{1}{R}\frac{\mathrm{d}\sigma}{\mathrm{d}T}\frac{\partial T}{\partial p_o} \;=\; 0 \tag{19}$$

$$\frac{1}{R}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}T}\frac{\partial T}{\partial T_o} \;-\; \frac{\mathrm{d}\sigma_o}{\mathrm{d}T_o}\right) \;=\; 0 \tag{20}$$

$$\frac{1}{R}\frac{\mathrm{d}\sigma}{\mathrm{d}T}\frac{\partial T}{\partial \pi} \;-\; \frac{1}{\eta_{\text{pol}}{}^{\pm 1}}\frac{1}{\pi} \;=\; 0 \tag{21}$$

Using $\mathrm{d}\sigma/\mathrm{d}T = c_{p(T)}/T$ these then give $\partial T/\partial(\;)$, and also give the remaining p and h derivatives via the chain rule.

$$\frac{\partial T}{\partial p_o} = 0 \tag{22}$$

$$\frac{\partial T}{\partial T_o} = \frac{c_{p_o}}{T_o}\frac{T}{c_{p(T)}} \tag{23}$$

$$\frac{\partial T}{\partial \pi} = \frac{R}{\pi\,\eta_{\mathrm{pol}}{}^{\pm 1}}\frac{T}{c_{p(T)}} \tag{24}$$

$$\frac{\partial p}{\partial p_o} = \pi \tag{25}$$

$$\frac{\partial p}{\partial T_o} = 0 \tag{26}$$

$$\frac{\partial p}{\partial \pi} = p_o \tag{27}$$

$$\frac{\partial h}{\partial p_o} = \frac{\mathrm{d}h}{\mathrm{d}T}\frac{\partial T}{\partial p_o} = 0 \tag{28}$$

$$\frac{\partial h}{\partial T_o} = \frac{\mathrm{d}h}{\mathrm{d}T}\frac{\partial T}{\partial T_o} = c_{p(T)}\frac{c_{p_o}}{T_o}\frac{T}{c_{p(T)}} = \frac{c_{p_o}}{T_o}T \tag{29}$$

$$\frac{\partial h}{\partial \pi} = \frac{\mathrm{d}h}{\mathrm{d}T}\frac{\partial T}{\partial \pi} = c_{p(T)}\frac{R}{\pi}\frac{T}{c_{p(T)}} = \frac{R}{\pi\,\eta_{\mathrm{pol}}{}^{\pm 1}}T \tag{30}$$

3.5 Pure loss prescribed

A pure loss with no work or heat addition is the limiting case of a prescribed pressure ratio $\pi < 1$, with $\eta_{\mathrm{pol}} = 0$. The relations above then greatly simplify to the following.

$$p = p_o\,\pi \tag{31}$$

$$T = T_o \tag{32}$$

$$h = h_o \tag{33}$$

3.6 Enthalpy difference prescribed

For prescribed $p_o, T_o, \Delta h, \eta_{\mathrm{pol}}$, the new state p, T, h after a compression/expansion process is computed as follows.

$$h_o = h(T_o) \tag{34}$$

$$\sigma_o = \sigma(T_o) \tag{35}$$

$$c_{p_o} = c_{p(T_o)} \tag{36}$$

initial guess: $$T = T_o + \Delta h/c_{p_o} \tag{37}$$

solve: $$\mathcal{R}_h(T\,;\,p_o, T_o, \Delta h) \equiv h(T) - h_o - \Delta h = 0 \quad\rightarrow\quad T \tag{38}$$

then: $$p = p_o \exp\!\left(\eta_{\mathrm{pol}}{}^{\pm 1}\,\frac{\sigma(T) - \sigma_o}{R}\right) \tag{39}$$

$$h = h(T) \tag{40}$$

The solution for T is via Newton iteration. The overall calculation will be denoted by

$$\{p, T, h, c_p, R\} \;=\; \mathcal{F}_h\left(\vec{\alpha}, p_o, T_o, \Delta h, {\eta_{\rm pol}}^{\pm 1}\right) \tag{41}$$

The function Jacobian derivatives $\partial\{p, T, h\}/\partial\{p_o, T_o, \Delta h\}$ are obtained by first implicitly differentiating the residual function (38) with respect to the specified $p_o, T_o, \Delta h$.

$$\frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial p_o} \;=\; 0 \tag{42}$$

$$\frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial T_o} - \frac{{\rm d}h_o}{{\rm d}T_o} \;=\; 0 \tag{43}$$

$$\frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial \Delta h} - 1 \;=\; 0 \tag{44}$$

Using ${\rm d}h/{\rm d}T = c_{p(T)}$ these then give $\partial T/\partial()$, and also give the remaining p and h derivatives via the chain rule.

$$\frac{\partial T}{\partial p_o} \;=\; 0 \tag{45}$$

$$\frac{\partial T}{\partial T_o} \;=\; \frac{c_{p_o}}{c_{p(T)}} \tag{46}$$

$$\frac{\partial T}{\partial \Delta h} \;=\; \frac{1}{c_{p(T)}} \tag{47}$$

$$\frac{\partial p}{\partial p_o} \;=\; \frac{p}{p_o} \;+\; p\,\frac{{\eta_{\rm pol}}^{\pm 1}}{R}\left(\frac{{\rm d}\sigma}{{\rm d}T}\frac{\partial T}{\partial p_o}\right) \;=\; \frac{p}{p_o} \tag{48}$$

$$\frac{\partial p}{\partial T_o} \;=\; p\,\frac{{\eta_{\rm pol}}^{\pm 1}}{R}\left(\frac{{\rm d}\sigma}{{\rm d}T}\frac{\partial T}{\partial T_o} - \frac{{\rm d}\sigma_o}{{\rm d}T_o}\right) \;=\; p\,\frac{{\eta_{\rm pol}}^{\pm 1}}{R}\left(\frac{c_{p(T)}}{T} - \frac{c_{p_o}}{T_o}\right) \tag{49}$$

$$\frac{\partial p}{\partial \Delta h} \;=\; p\,\frac{{\eta_{\rm pol}}^{\pm 1}}{R}\frac{{\rm d}\sigma}{{\rm d}T}\frac{\partial T}{\partial \Delta h} \;=\; p\,\frac{{\eta_{\rm pol}}^{\pm 1}}{R}\frac{1}{T} \tag{50}$$

$$\frac{\partial h}{\partial p_o} \;=\; \frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial p_o} \;=\; 0 \tag{51}$$

$$\frac{\partial h}{\partial T_o} \;=\; \frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial T_o} \;=\; c_{p(T)}\,\frac{c_{p_o}}{c_{p(T)}} \;=\; c_{p_o} \tag{52}$$

$$\frac{\partial h}{\partial \Delta h} \;=\; \frac{{\rm d}h}{{\rm d}T}\frac{\partial T}{\partial \Delta h} \;=\; c_{p(T)}\,\frac{1}{c_{p(T)}} \;=\; 1 \tag{53}$$

3.7 Composition change prescribed

A composition change, such as due to combustion, is specified by the following mass fractions and input properties, with γ_i including the effect of the combustor efficiency $\eta_{\rm b}$.

α_i	constituent i mass fraction for air
β_i	constituent i mass fraction for fuel vapor
γ_i	constituent i mass fraction change in air due to combustion
T_o	air temperature before combustion
$T_{\rm fuel}$	fuel vapor temperature before combustion
T	temperature after combustion

The following quantities are computed:

$\bar{f}$	fuel/combustor-air mass ratio
λ_i	constituent i mass fraction for combustion products

$$\vec{h}_o = h_{i(T_o)} \tag{54}$$
$$\vec{h} = h_{i(T)} \tag{55}$$
$$\vec{h}_{\text{fuel}} = h_{i(T_{\text{fuel}})} \tag{56}$$

Defining the combustor air mass flow $\dot{m}_{\text{air}} \equiv \dot{m}_{\text{core}} - \dot{m}_{\text{offt}}$, the enthalpy balance across the combustor is

$$\dot{m}_{\text{air}}\,\vec{\alpha}\cdot\vec{h}_o \;+\; \dot{m}_{\text{fuel}}\,\vec{\beta}\cdot\vec{h}_{\text{fuel}} \;=\; \dot{m}_{\text{air}}\,\vec{\alpha}\cdot\vec{h} \;+\; \dot{m}_{\text{fuel}}\,\vec{\gamma}\cdot\vec{h} \tag{57}$$

which is rearranged to give the fuel/air mass ratio.

$$\bar{f} \;\equiv\; \frac{\dot{m}_{\text{fuel}}}{\dot{m}_{\text{air}}} \;=\; \frac{\vec{\alpha}\cdot\vec{h} \,-\, \vec{\alpha}\cdot\vec{h}_o}{\vec{\beta}\cdot\vec{h}_{\text{fuel}} \,-\, \vec{\gamma}\cdot\vec{h}} \tag{58}$$

The mass fraction vector $\vec{\lambda}$ of the combustion products is subsequently obtained from the mass balance across the combustor.

$$(\dot{m}_{\text{air}} + \dot{m}_{\text{fuel}})\,\vec{\lambda} \;=\; \dot{m}_{\text{air}}\,\vec{\alpha} \;+\; \dot{m}_{\text{fuel}}\,\vec{\gamma} \tag{59}$$
$$\vec{\lambda} \;=\; \frac{\vec{\alpha} + \bar{f}\vec{\gamma}}{1+\bar{f}} \tag{60}$$

The overall combustion-change calculation will be denoted by

$$\left\{\bar{f},\vec{\lambda}\right\} \;=\; \mathcal{F}_b\left(\vec{\alpha},\vec{\beta},\vec{\gamma},\eta_{\text{b}},T_o,T_{\text{fuel}},T\right) \tag{61}$$

The Jacobian derivatives of $\bar{f}$ and $\vec{\lambda}$ are obtained by direct differentiation of their definitions (58) and (60).

$$\frac{\partial\bar{f}}{\partial T_o} \;=\; -\frac{\vec{\alpha}\cdot\vec{c}_{p_o}}{\vec{\beta}\cdot\vec{h}_{\text{fuel}} \,-\, \vec{\gamma}\cdot\vec{h}} \tag{62}$$
$$\frac{\partial\bar{f}}{\partial T_{\text{fuel}}} \;=\; -\bar{f}\,\frac{\vec{\beta}\cdot\vec{c}_{p_{\text{fuel}}}}{\vec{\beta}\cdot\vec{h}_{\text{fuel}} \,-\, \vec{\gamma}\cdot\vec{h}} \tag{63}$$
$$\frac{\partial\bar{f}}{\partial T} \;=\; \frac{\vec{\alpha}\cdot\vec{c}_p}{\vec{\beta}\cdot\vec{h}_{\text{fuel}} \,-\, \vec{\gamma}\cdot\vec{h}} \tag{64}$$

$$\frac{\partial\vec{\lambda}}{\partial\bar{f}} \;=\; \frac{\vec{\gamma}-\vec{\lambda}}{1+\bar{f}} \tag{65}$$
$$\frac{\partial\vec{\lambda}}{\partial T_o} \;=\; \frac{\partial\vec{\lambda}}{\partial\bar{f}}\,\frac{\partial\bar{f}}{\partial T_o} \tag{66}$$
$$\frac{\partial\vec{\lambda}}{\partial T_{\text{fuel}}} \;=\; \frac{\partial\vec{\lambda}}{\partial\bar{f}}\,\frac{\partial\bar{f}}{\partial T_{\text{fuel}}} \tag{67}$$
$$\frac{\partial\vec{\lambda}}{\partial T} \;=\; \frac{\partial\vec{\lambda}}{\partial\bar{f}}\,\frac{\partial\bar{f}}{\partial T} \tag{68}$$

3.8 Mixing

Mixing between two streams is a simplified version of the combustion case above. No chemical reaction is assumed, so that $\vec{\gamma} = 0$. However, in general the two streams will have two different chemical compositions specified by their mass fraction vectors $\vec{\lambda}_a$ and $\vec{\lambda}_b$, two different temperatures T_a and T_b, and two different enthalpies $\vec{h}_a = h_{i(T_a)}$ and $\vec{h}_b = h_{i(T_b)}$. The species mass flow balance gives the composition mass fraction vector $\vec{\lambda}$ of the mixed gas, in terms of the convenient relative mass fractions f_a, f_b of the two streams.

$$f_a = \frac{\dot{m}_a}{\dot{m}_a + \dot{m}_b} \tag{69}$$

$$f_b = \frac{\dot{m}_b}{\dot{m}_a + \dot{m}_b} \tag{70}$$

$$\vec{\lambda} = f_a \vec{\lambda}_a + f_b \vec{\lambda}_b \tag{71}$$

Without any chemical reaction change term, the mixed enthalpy is

$$\vec{\lambda} \cdot \vec{h}_{(T)} \equiv h_{\text{mix}} = f_a \vec{\lambda}_a \cdot \vec{h}_a + f_b \vec{\lambda}_b \cdot \vec{h}_b \tag{72}$$

which can be numerically inverted for the mixed temperature T, using the previously-defined $\mathcal{F}_T$ function.

$$T_{\text{guess}} = f_a T_a + f_b T_b \tag{73}$$

$$T = \mathcal{F}_T \left(\vec{\lambda}, h_{\text{mix}} \, ; \, T_{\text{guess}} \right) \tag{74}$$

3.9 Mach number prescribed

For prescribed $p_o, T_o, M_o, M, \eta_{\text{pol}}$, the new adiabatic-change state p, T, h corresponding to M is computed as follows.

$$\sigma_o = \sigma_{(T_o)} \tag{75}$$

$$c_{p_o} = c_{p(T_o)} \tag{76}$$

$$h_o = h_{(T_o)} \tag{77}$$

$$u_o^2 = M_o^2 \frac{c_{p_o} R_o}{c_{p_o} - R_o} T_o \tag{78}$$

initial guess:

$$T = T_o \frac{1 + \frac{R_o}{2(c_{p_o} - R_o)} M_o^2}{1 + \frac{R_o}{2(c_{p_o} - R_o)} M^2} \tag{79}$$

solve:

$$\mathcal{R}_M(T \, ; \, p_o, T_o, M_o, M) \equiv h_{(T)} + \frac{1}{2} M^2 \frac{c_{p(T)} R}{c_{p(T)} - R} T - h_o - \frac{1}{2} u_o^2 = 0 \quad \rightarrow \quad T \tag{80}$$

then:

$$p = p_o \exp\left({\eta_{\text{pol}}}^{\pm 1} \frac{\sigma_{(T)} - \sigma_o}{R} \right) \tag{81}$$

$$h = h_{(T)} \tag{82}$$

The solution for T is via Newton iteration. The overall calculation will be denoted by

$$\{p, T, h, c_p, R\} = \mathcal{F}_M \left(\vec{\alpha}, p_o, T_o, M_o, M, {\eta_{\text{pol}}}^{\pm 1} \right) \tag{83}$$

The convenient u^2 and u_o^2 derivatives are defined next.

$$u^2 = M^2 \frac{c_{p(T)}\, R}{c_{p(T)} - R}\, T \tag{84}$$

$$\frac{\partial u^2}{\partial M} = 2M \frac{c_{p(T)}\, R}{c_{p(T)} - R}\, T \tag{85}$$

$$\frac{\partial u^2}{\partial T} = M^2 \frac{R}{c_{p(T)} - R} \left(c_{p(T)} - \frac{R}{c_{p(T)} - R}\, c'_{p(T)}\, T \right) \tag{86}$$

$$\frac{\partial u_o^2}{\partial M_o} = 2M_o \frac{c_{p_o}\, R_o}{c_{p_o} - R_o}\, T_o \tag{87}$$

$$\frac{\partial u_o^2}{\partial T_o} = M_o^2 \frac{R_o}{c_{p_o} - R_o} \left(c_{p_o} - \frac{R_o}{c_{p_o} - R_o}\, c'_{p_o}\, T_o \right) \tag{88}$$

The function Jacobian derivatives $\partial \{p, T, h\} / \partial \{p_o, T_o, M_o, M\}$ are then obtained by first implicitly differentiating (80) with respect to the specified p_o, T_o, M_o, M.

$$\frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial p_o} + \frac{1}{2} \frac{\partial u^2}{\partial T} \frac{\partial T}{\partial p_o} = 0 \tag{89}$$

$$\frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial T_o} + \frac{1}{2} \frac{\partial u^2}{\partial T} \frac{\partial T}{\partial T_o} - \frac{\partial h_o}{\partial T_o} - \frac{\partial u_o^2}{\partial T_o} = 0 \tag{90}$$

$$\frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial M_o} + \frac{1}{2} \frac{\partial u^2}{\partial T} \frac{\partial T}{\partial M_o} - \frac{\partial u_o^2}{\partial M_o} = 0 \tag{91}$$

$$\frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial M} + \frac{1}{2} \frac{\partial u^2}{\partial T} \frac{\partial T}{\partial M} = 0 \tag{92}$$

Using $\mathrm{d}h/\mathrm{d}T = c_{p(T)}$ these then give $\partial T/\partial()$, and also give the remaining p and h derivatives via the chain rule.

$$\frac{\partial T}{\partial p_o} = 0 \tag{93}$$

$$\frac{\partial T}{\partial T_o} = \frac{c_{p_o}}{c_{p(T)}} \tag{94}$$

$$\frac{\partial T}{\partial \Delta h} = \frac{1}{c_{p(T)}} \tag{95}$$

$$\frac{\partial p}{\partial p_o} = \frac{p}{p_o} + p \frac{{\eta_{\mathrm{pol}}}^{\pm 1}}{R} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}T} \frac{\partial T}{\partial p_o} \right) = \frac{p}{p_o} \tag{96}$$

$$\frac{\partial p}{\partial T_o} = p \frac{{\eta_{\mathrm{pol}}}^{\pm 1}}{R} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}T} \frac{\partial T}{\partial T_o} - \frac{\mathrm{d}\sigma_o}{\mathrm{d}T_o} \right) = p \frac{{\eta_{\mathrm{pol}}}^{\pm 1}}{R} \left(\frac{c_{p(T)}}{T} - \frac{c_{p_o}}{T_o} \right) \tag{97}$$

$$\frac{\partial p}{\partial \Delta h} = p \frac{{\eta_{\mathrm{pol}}}^{\pm 1}}{R} \frac{\mathrm{d}\sigma}{\mathrm{d}T} \frac{\partial T}{\partial \Delta h} = p \frac{{\eta_{\mathrm{pol}}}^{\pm 1}}{R} \frac{1}{T} \tag{98}$$

$$\frac{\partial h}{\partial p_o} = \frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial p_o} = 0 \tag{99}$$

$$\frac{\partial h}{\partial T_o} = \frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial T_o} = c_{p(T)} \frac{c_{p_o}}{c_{p(T)}} = c_{p_o} \tag{100}$$

$$\frac{\partial h}{\partial \Delta h} = \frac{\mathrm{d}h}{\mathrm{d}T} \frac{\partial T}{\partial \Delta h} = c_{p(T)} \frac{1}{c_{p(T)}} = 1 \tag{101}$$

3.10 Mass flux prescribed

It is occasionally useful to calculate the static quantities p, T, h corresponding to a specified stagnation state p_o, T_o, h_o, and a specified mass flux $\rho u = \dot{m}/A \equiv m'$. This is computed as follows, starting from some given initial guess specified by the Mach number M_{guess}, which also selects the subsonic or supersonic branch.

$$\sigma_o \;=\; \sigma(T_o) \tag{102}$$

$$h_o \;=\; h(T_o) \tag{103}$$

$$\text{initial guess:} \qquad T \;=\; T_o \left(1 + \frac{R_o}{2(c_{p_o} - R_o)} M_{\text{guess}}^2 \right)^{-1} \tag{104}$$

$$\text{solve:} \qquad \mathcal{R}_m(T\,;\,p_o, T_o, m') \;\equiv\; \left(\frac{p(T)}{RT}\right)^2 2\Big(h_o - h(T)\Big) \;-\; (m')^2 \;=\; 0 \quad \rightarrow \quad T \tag{105}$$

$$\text{then:} \qquad p \;=\; p_o \exp\!\left(\frac{\sigma(T) - \sigma_o}{R}\right) \tag{106}$$

$$h \;=\; h(T) \tag{107}$$

The solution for T is via Newton iteration. The overall calculation will be denoted by

$$\{p, T, h, c_p, R\} \;=\; \mathcal{F}_m\,(\vec{\alpha}, p_o, T_o, m'\,;\,M_{\text{guess}}) \tag{108}$$

This function's Jacobian derivatives can be calculated by the same procedures used for the other functions.

4 Turbofan Component Calculations

The assumed turbofan engine configuration is shown in Figure 1. Gas state variables and velocities are defined or computed at all the numbered locations.

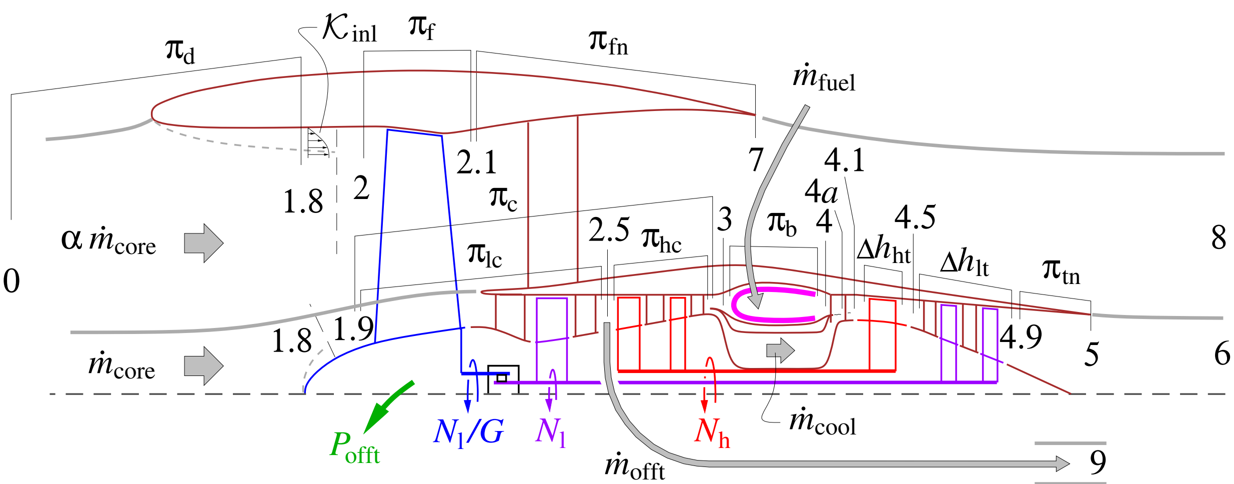

Figure 1: Engine station numbers, mass flows, total-pressure ratios, total-enthalpy drops, and spool speeds.

Most of the calculations described in this section are common to both the design and the off-design cases, and assume that the quantities listed in Table 1 are known. The simple design case without offtakes and without an inlet boundary layer defect will require only a single calculation pass, with the core mass flow and component dimensions determined at the end. In contrast, the design case with offtakes and the off-design case requires multiple Newton-iteration passes to converge the component pressure ratios and mass flows.

Table 1: Quantities assumed known for subsequent turbofan component computations. Their values are either specified from the flight condition, known as design-case quantities, or known from a previous iteration.

T_0, p_0	atmospheric properties
M_0	flight Mach number
$T_{t\,4}$	combustor exit total temperature
$\pi_{\rm f}$	fan pressure ratio
$\pi_{\rm lc}$	LPC pressure ratio
$\pi_{\rm hc}$	HPC pressure ratio
α	fan bypass ratio
$\pi_{\rm d}$	diffuser pressure ratio
$\pi_{\rm b}$	combustor pressure ratio
$\pi_{\rm fn}$	fan duct loss pressure ratio
M_{4a}	representative Mach number at start of HPT cooling-flow mixing zone
$T_{\rm m}$	HPT design metal temperature (if $f_{\rm c}$ is to be sized)
$f_{\rm c}$	cooling mass flow fraction (if previously sized)
$\dot{m}_{\rm offt}$	LPC mass flow offtake
$P_{\rm offt}$	LPC spool power offtake
$\mathcal{K}_{\rm inl}$	inlet kinetic energy defect
$\dot{m}_{\rm core}$	core mass flow (not needed if offtakes and $\mathcal{K}_{\rm inl}$ are ignored)
$(\,)_{\rm D}$	specified design-case quantity

4.1 Freestream properties

From the specified freestream static temperature, pressure, and Mach number, T_0, p_0, M_0, we can obtain the freesteam speed of sound and velocity.

$$c_{p_0} = c_{p(T_0)} \tag{109}$$

$$a_0 = \sqrt{\frac{c_{p_0}}{c_{p_0} - R_0} \, R_0 T_0} \tag{110}$$

$$u_0 = M_0 \, a_0 \tag{111}$$

4.2 Freestream-stagnation properties

The freestream stagnation quantities are computed using the specified enthalpy change procedure, with $\eta_{\rm pol} = 1$.

$$\Delta h = \tfrac{1}{2} u_0^2 \tag{112}$$

$$\{p_{t\,0}, T_{t\,0}, h_{t\,0}\} = \mathcal{F}_h \left(\vec{\alpha}, p_0, T_0, \Delta h, 1\right) \tag{113}$$

The standard fixed-c_p relations could also be used here, since the stagnation-static temperature difference is sufficiently small for any non-hypersonic flight Mach number.

4.3 Fan and compressor quantities

4.3.1 Inlet conditions

The stagnation conditions $()_{t\,1.8}$ in the inlet inviscid flow (excluding the inlet BLs) are computed as the pure-loss case with a diffuser total/total pressure ratio π_d.

$$p_{t\,1.8} = p_{t\,0}\,\pi_\mathrm{d} \tag{114}$$

$$T_{t\,1.8} = T_{t\,0} \tag{115}$$

$$h_{t\,1.8} = h_{t\,0} \tag{116}$$

Normally $\pi_\mathrm{d} \simeq 1$, unless an inlet screen or other losses are present upstream.

4.3.2 Fan and LPC inlet conditions

In the case of an ingested boundary layer, the additional total pressure loss at station 2 is defined in terms of the static temperature and pressure profiles $T_{(n)}$, $p_{(n)}$, and the corresponding entropy profile $s_{(n)}$,

$$\begin{aligned} s_{(n)} &\equiv \ln \frac{(T_{(n)}/T_{t\,1.8})^{\gamma/(\gamma-1)}}{p_{(n)}/p_{t\,1.8}} \\ &= \ln \frac{(T_{(n)}/T_e)^{\gamma/(\gamma-1)}}{p_{(n)}/p_e} + \ln \frac{(T_e/T_{t\,1.8})^{\gamma/(\gamma-1)}}{p_e/p_{t\,1.8}} && (117) \\ &\simeq \frac{\gamma}{\gamma-1} \ln \frac{T_{(n)}}{T_e} && (118) \\ &= \frac{\gamma}{\gamma-1}\frac{\Delta T}{T_e} + \mathcal{O}\left\{\left(\frac{\Delta T}{T_e}\right)^2\right\} && (119) \end{aligned}$$

where T_e and p_e are the values just outside the boundary layer edge at station 2.

$$T_e = T_{t\,1.8}\left(1 + \tfrac{\gamma-1}{2} M_2^2\right)^{-1} \tag{120}$$

$$p_e = p_{t\,1.8}\left(1 + \tfrac{\gamma-1}{2} M_2^2\right)^{-\gamma/(\gamma-1)} \tag{121}$$

We've assumed that these have the same stagnation state as station 1.8, which made the second term in equation (117) vanish. To then obtain (118) we've also assumed the boundary layer approximation $p/p_e \simeq 1$. Finally, equation (119) is a Taylor series for the logarithm in terms of the fractional temperature defect, which for a uniform total temperature (correct in a mass-averaged sense), is

$$\frac{\Delta T_{(n)}}{T_e} \equiv \frac{T_{(n)} - T_e}{T_e} = \frac{u_e^2 - u^2}{2\,c_{p_{1.8}} T_e} = \frac{\gamma-1}{2}\frac{u_e^2 - u^2}{a_e^2} \tag{122}$$

$$a_e^2 = (\gamma-1)\,c_{p_{1.8}} T_e = (\gamma-1)\,c_{p_{1.8}} T_{t\,1.8}\left(1 + \tfrac{\gamma-1}{2} M_2^2\right)^{-1} \tag{123}$$

so that the approximate entropy profile (119) simplifies as follows.

$$s_{(n)} \simeq \frac{\gamma}{2}\frac{u_e^2 - u^2}{a_e^2} \tag{124}$$

We now define the corresponding mass-averaged entropy $\bar{s}$,

$$\dot{m}_{\mathrm{mix}} \equiv \iint \rho u \, \mathrm{d}A_{\mathrm{inl}} \tag{125}$$

$$\bar{s} \equiv \frac{1}{\dot{m}_{\mathrm{mix}}} \iint s \, \rho u \, \mathrm{d}A_{\mathrm{inl}} \simeq \frac{\mathcal{K}_{\mathrm{inl}}}{\dot{m}_{\mathrm{mix}}} \frac{\gamma}{a_e^2} \tag{126}$$

$$\mathcal{K}_{\mathrm{inl}} \equiv \iint \tfrac{1}{2} \left(u_e^2 - u^2 \right) \rho u \; \mathrm{d}A \tag{127}$$

where $\mathcal{K}_{\mathrm{inl}}$ is the kinetic energy defect. Possible assumptions for the mixing mass flow are

$$\dot{m}_{\mathrm{mix}} = \begin{cases} \dot{m}_{\mathrm{fan}} & , \text{ BL mixes only with the fan flow} \\ \dot{m}_{\mathrm{fan}} + \dot{m}_{\mathrm{core}} & , \text{ BL mixes with entire inlet flow} \end{cases} \tag{128}$$

$$\dot{m}_{\mathrm{fan}} \simeq \bar{m}_{\mathrm{f}} \sqrt{\frac{T_{\mathrm{ref}}}{T_{t\,0}}} \frac{p_{t\,0}}{p_{\mathrm{ref}}} \tag{129}$$

$$\dot{m}_{\mathrm{core}} \simeq \bar{m}_{\mathrm{lc}} \sqrt{\frac{T_{\mathrm{ref}}}{T_{t\,0}}} \frac{p_{t\,0}}{p_{\mathrm{ref}}} \tag{130}$$

and the fan inlet state at station 2 are then

$$p_{t\,2} = p_{t\,1.8} \exp(-\bar{s}) \tag{131}$$

$$T_{t\,2} = T_{t\,1.8} \tag{132}$$

$$h_{t\,2} = h_{t\,1.8} \tag{133}$$

and the corresponding LPC inlet state at station 1.9 is

$$(\,)_{t\,1.9} = \begin{cases} (\,)_{t\,1.8} & , \text{ BL mixes only with the fan flow} \\ (\,)_{t\,2} & , \text{ BL mixes with entire inlet flow} \end{cases} \tag{134}$$

The fan and core mass flow approximations (129) and (130) use the freestream total quantities $(\,)_{t\,0}$ instead of $(\,)_{t\,2}$ and $(\,)_{t\,1.9}$ called for by the $\bar{m}_{\mathrm{f}}$ and $\bar{m}_{\mathrm{lc}}$ definitions. This is done to avoid making the $p_{t\,2}$ and $p_{t\,1.9}$ definitions above circular, which would then need sub-iterations to resolve. The physical assumptions inherent in the mix-out state calculation justify this relatively minor approximation and simplification.

4.3.3 Fan exit conditions

The fan exit stagnation conditions are computed from the fan pressure ratio π_{f}. The polytropic efficiency is computed first using the appropriate assumed fan efficiency map function $\mathcal{F}_\eta$ given in section 7.2.

$$\eta_{\mathrm{pol_f}} = \begin{cases} \mathcal{F}_\eta(\pi_{\mathrm{f}}\,,\ \pi_{\mathrm{f}}\ ,\ 1\ ,\ 1\,) & , \text{ (design case)} \\ \mathcal{F}_\eta(\pi_{\mathrm{f}}\,, \pi_{\mathrm{f\,D}}\,, \bar{m}_{\mathrm{f}}, \bar{m}_{\mathrm{f\,D}}) & , \text{ (off-design case)} \end{cases} \tag{135}$$

$$\{p_{t\,2.1}, T_{t\,2.1}, h_{t\,2.1}\} = \mathcal{F}_p \left(\vec{\alpha}, p_{t\,2}, T_{t\,2}, \pi_{\mathrm{f}}, \eta_{\mathrm{pol_f}} \right) \tag{136}$$

4.3.4 Fan nozzle exit conditions

Fan duct and fan nozzle losses are represented by the total pressure-drop ratio π_{fn} with no total enthalpy change.

$$p_{t\,7} = p_{t\,2.1}\, \pi_{\mathrm{fn}} \tag{137}$$

$$T_{t\,7} = T_{t\,2.1} \tag{138}$$

$$h_{t\,7} = h_{t\,2.1} \tag{139}$$

4.3.5 LPC exit conditions

The LPC calculation is the same as for the fan, but the inlet state is 1.9 and the pressure ratio and efficiency are of course different.

$$\eta_{\mathrm{pol}_{\mathrm{lc}}} = \begin{cases} \mathcal{F}_\eta(\pi_{\mathrm{lc}}\,,\ \pi_{\mathrm{lc}}\ ,\ 1\ ,\ 1\,) & ,\ \text{(design case)} \\ \mathcal{F}_\eta(\pi_{\mathrm{lc}}, \pi_{\mathrm{lcD}}\,, \bar{m}_{\mathrm{lc}}, \bar{m}_{\mathrm{lcD}}) & ,\ \text{(off-design case)} \end{cases} \tag{140}$$

$$\{p_{t\,2.5}, T_{t\,2.5}, h_{t\,2.5}\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,1.9}, T_{t\,1.9}, \pi_{\mathrm{lc}}, \eta_{\mathrm{pol}_{\mathrm{lc}}}\right) \tag{141}$$

4.3.6 HPC exit conditions

The HPC calculation procedure is the same as for the LPC and fan, except that the 2.5 station quantities from the LPC calculation above are used for the HPC inlet.

$$\eta_{\mathrm{pol}_{\mathrm{hc}}} = \begin{cases} \mathcal{F}_\eta(\pi_{\mathrm{hc}}\,,\ \pi_{\mathrm{hc}}\ ,\ 1\ ,\ 1\,) & ,\ \text{(design case)} \\ \mathcal{F}_\eta(\pi_{\mathrm{hc}}, \pi_{\mathrm{hcD}}\,, \bar{m}_{\mathrm{hc}}, \bar{m}_{\mathrm{hcD}}) & ,\ \text{(off-design case)} \end{cases} \tag{142}$$

$$\{p_{t\,3}, T_{t\,3}, h_{t\,3}\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,2.5}, T_{t\,2.5}, \pi_{\mathrm{hc}}, \eta_{\mathrm{pol}_{\mathrm{hc}}}\right) \tag{143}$$

4.3.7 Fan and compressor efficiencies

The equivalent isentropic states and overall efficiencies can be computed for the fan and compressors out of interest, although these are not required for any subsequent calculations.

$$\{p_{t\,2.1}, (T_{t\,2.1})_{is}, (h_{t\,2.1})_{is}\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,2}, T_{t\,2}, \pi_{\mathrm{f}}, 1\right) \tag{144}$$

$$\{p_{t\,2.5}, (T_{t\,2.5})_{is}, (h_{t\,2.5})_{is}\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,1.9}, T_{t\,1.9}, \pi_{\mathrm{lc}}, 1\right) \tag{145}$$

$$\{p_{t\,3}, (T_{t\,3})_{is}, (h_{t\,3})_{is}\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,2.5}, T_{t\,2.5}, \pi_{\mathrm{hc}}, 1\right) \tag{146}$$

$$\eta_{\mathrm{f}} = \frac{(h_{t\,2.1})_{is} - h_{t\,2}}{h_{t\,2.1} - h_{t\,2}} \tag{147}$$

$$\eta_{\mathrm{lc}} = \frac{(h_{t\,2.5})_{is} - h_{t\,1.9}}{h_{t\,2.5} - h_{t\,1.9}} \tag{148}$$

$$\eta_{\mathrm{hc}} = \frac{(h_{t\,3})_{is} - h_{t\,2.5}}{h_{t\,3} - h_{t\,2.5}} \tag{149}$$

4.4 Cooling Mass Flow or Metal Temperature Calculations

Cooling-flow calculations consist of either

1) Determination of cooling mass flow ratio (cooling sizing), or
2) Determination of metal temperature (cooling analysis).

It should be noted that the cooling sizing case 1) may be performed for any operating point, and not necessarily the engine-sizing design point. For example, an engine whose design sizing case is the cruise condition will typically have its cooling flow ratio sized at the off-design takeoff condition.

4.4.1 Cooling Mass Flow Ratio Sizing

For the cooling-sizing case, the cooling mass flow fractions ε_1, $\varepsilon_2 \ldots$ for the hot-section blade rows are determined to obtain required blade-row metal temperatures $T_{\mathrm{m}_1}, T_{\mathrm{m}_2} \ldots$ as described in the document *Film Cooling Flow Loss Model*. The function has the form

$$\{\varepsilon_1, \varepsilon_2, \ldots\} \;=\; \mathcal{F}_\varepsilon \left(T_{t3}, T_{t4}, T_{\mathrm{m}_1}, T_{\mathrm{m}_2} \ldots \,;\, M_{\mathrm{exit}}, \Delta T_{\mathrm{streak}}, St_A, \theta_f, \eta\right) \tag{150}$$

where $M_{\mathrm{exit}} \ldots \eta$ are the various parameters in the cooling model. The overall cooling mass flow is the sum of the individual blade-row cooling mass flows. Since the compressor discharge is reduced by any compressor mass flow offtake farther upstream, we rescale the blade-row fractions by $(\dot{m}_{\mathrm{core}} - \dot{m}_{\mathrm{offt}})/\dot{m}_{\mathrm{core}} = 1 - f_{\mathrm{o}}$ to give the overall cooling flow fraction f_{c} which is defined relative to the inlet core mass flow.

$$f_{\mathrm{o}} \;\equiv\; \dot{m}_{\mathrm{offt}}/\dot{m}_{\mathrm{core}} \tag{151}$$

$$f_{\mathrm{c}} \;\equiv\; \dot{m}_{\mathrm{cool}}/\dot{m}_{\mathrm{core}} \;=\; (1 - f_{\mathrm{o}})\,(\varepsilon_1 + \varepsilon_2 + \ldots) \tag{152}$$

4.4.2 Metal Temperature Calculation

In this case the individual blade-row cooling mass flow ratios $\varepsilon_1, \varepsilon_2 \ldots$ are assumed to be known. The blade-row metal temperatures can then be determined from the cooling model relations, which are now recast into the following form.

$$\{T_{\mathrm{m}_1}, T_{\mathrm{m}_2}, \ldots\} \;=\; \mathcal{F}_{T_\mathrm{m}} \left(T_{t3}, T_{t4}, \varepsilon_1, \varepsilon_2 \ldots \,;\, M_{\mathrm{exit}}, \Delta T_{\mathrm{streak}}, St_A, \theta_f, \eta\right) \tag{153}$$

These metal temperatures are only informative, since they are not required for any subsequent calculations.

4.5 Combustor quantities

The engine possibly has cooling air, which is bled from station 3, bypasses the combustor, and re-enters the flow path in the HPT, as sketched in Figure 1. The mass flow fractions and control volumes are detailed in Figure 2.

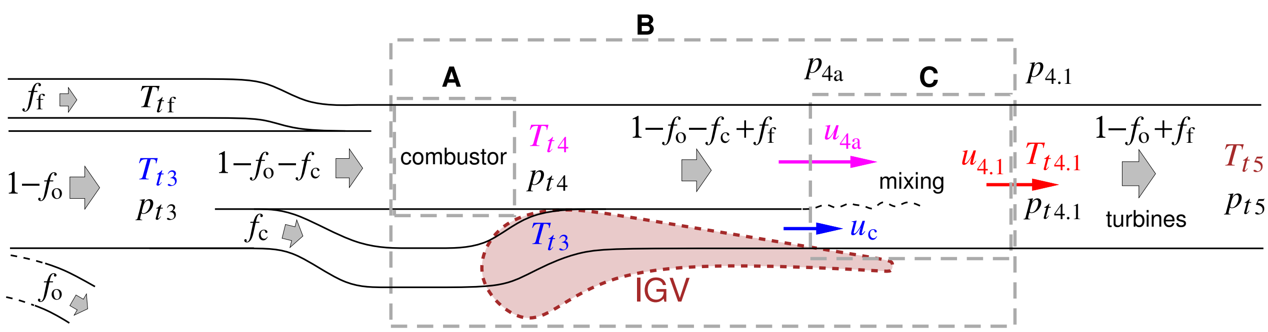

Figure 2: Combustor and film-cooling flows, with mixing over and downstream of the IGV. Dashed rectangles are control volumes. Mass flow fractions are relative to $\dot{m}_{\mathrm{core}}$, upstream of the mass offtake fraction f_{o} bleed location.

Using control volume A in Figure 2, the fuel/combustor-air mass flow fraction

$$\bar{f} \;\equiv\; \frac{\dot{m}_{\mathrm{fuel}}}{\dot{m}_{\mathrm{air}}} \;=\; \frac{\dot{m}_{\mathrm{fuel}}}{\dot{m}_{\mathrm{core}} - \dot{m}_{\mathrm{offt}} - \dot{m}_{\mathrm{cool}}} \;=\; \frac{f_{\mathrm{f}}}{1 - f_{\mathrm{o}} - f_{\mathrm{c}}} \tag{154}$$

and the combustion-product constituent mass fraction vector $\vec{\lambda}$ are obtained by using the compressor exit condition $(\,)_{t3}$, together with the specified combustor exit total temperature

$T_{t\,4}$. The fuel/core-air fraction f_{f} then follows by rescaling $\bar{f}$ by the combustor-air/core-air mass flow ratio $1-f_{\mathrm{o}}-f_{\mathrm{c}}$.

$$\left\{\bar{f}, \vec{\lambda}\right\} = \mathcal{F}_b\left(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \eta_{\mathrm{b}}, T_{t\,3}, T_{t\,\mathrm{f}}, T_{t\,4}\right) \tag{155}$$

$$f_{\mathrm{f}} = (1-f_{\mathrm{o}}-f_{\mathrm{c}})\,\bar{f} \tag{156}$$

The combustor exit conditions are then obtained using this $\vec{\lambda}$, together with the specified $T_{t\,4}$ and the assumed combustor pressure ratio π_{b}.

$$h_{t\,4} = \vec{\lambda}\cdot\vec{h}_{(T_{t\,4})} \tag{157}$$

$$\sigma_{t\,4} = \vec{\lambda}\cdot\vec{\sigma}_{(T_{t\,4})} \tag{158}$$

$$p_{t\,4} = p_{t\,3}\,\pi_{\mathrm{b}} \tag{159}$$

The heat release from the combustion is implicitly determined by the $\vec{\lambda}$ vector computed by the combustor function (155), together with the resulting enthalpy $h_{t\,4}$ defined by equation (157). The implied effective heating value of the fuel, which can be defined as

$$c_{p\,3} = \alpha_i\cdot\vec{c}_{p(T_{t\,3})} \tag{160}$$

$$c_{p\,4} = \lambda_i\cdot\vec{c}_{p(T_{t\,4})} \tag{161}$$

$$h_{\mathrm{fuel}} = \frac{c_{p\,3}+c_{p\,4}}{2}\,\frac{T_{t\,4}-T_{t\,3}+\bar{f}\,(T_{t\,4}-T_{t\,\mathrm{f}})}{\eta_{\mathrm{b}}\,\bar{f}} \tag{162}$$

is informative, although it should be noted that h_{fuel} is not used in any calculations. It also must not be confused with the fuel constituent enthalpy vector $\vec{h}_{\mathrm{fuel}}$.

4.6 Station 4.1 without IGV Cooling Flow

Without cooling flow ($f_{\mathrm{c}} = 0$), the 4.1 station quantities and constituent mass fraction at the first turbine rotor inlet are the same as the 4 station quantities at the combustor exit.

$$T_{t\,4.1} = T_{t\,4} \tag{163}$$

$$p_{t\,4.1} = p_{t\,4} \tag{164}$$

$$\vec{\lambda}' = \vec{\lambda} \tag{165}$$

The analysis can then skip the cooling flow mixing calculations below, and proceed directly to the **Turbine Quantities** section.

4.7 Station 4.1 with IGV Cooling Flow

The IGV pressure at the cooling flow exit is specified indirectly via the cooling-exit Mach M_{4a} at the combustor-exit stagnation conditions. The corresponding static conditions and velocity are calculated using the $\mathcal{F}_M$ function.

$$\{\,p_{4a}, T_{4a}, h_{4a}\,\} = \mathcal{F}_M\left(\vec{\lambda}, p_{t\,4}, T_{t\,4}, 0, M_{4a}, 1\right) \tag{166}$$

$$u_{4a} = \sqrt{2\,(h_{t\,4}-h_{4a})} \tag{167}$$

The cooling flow is assumed to exit the IGV at some fraction r_{u_c} of this u_{4a}.

$$u_{\rm c} \;=\; r_{u_c}\, u_{4a} \tag{168}$$

The combustor and cooling flows are assumed to be fully mixed at station 4.1. The mixed-out mass fraction vector $\vec{\lambda}'$ is calculated by the mass flow balance,

$$\vec{\lambda}' \;=\; \frac{1 - f_{\rm o} - f_{\rm c} + f_{\rm f}}{1 - f_{\rm o} + f_{\rm f}}\, \vec{\lambda} \;+\; \frac{f_{\rm c}}{1 - f_{\rm o} + f_{\rm f}}\, \vec{\alpha} \tag{169}$$

and is used for the downstream turbine and core exhaust calculations. Assuming a constant static pressure over the mixing region, a streamwise momentum balance gives the mixed-out velocity $u_{4.1}$.

$$p_{4.1} \;=\; p_{4a} \tag{170}$$

$$u_{4.1} \;=\; \frac{1 - f_{\rm o} - f_{\rm c} + f_{\rm f}}{1 - f_{\rm o} + f_{\rm f}}\, u_{4a} \;+\; \frac{f_{\rm c}}{1 - f_{\rm o} + f_{\rm f}}\, u_c \tag{171}$$

The enthalpy balance across control volume B in Figure 2 gives the mixed-out total temperature $T_{t\,4.1}$ via the $\mathcal{F}_h$ function.

$$h_{(T_{t\,4.1})} \;\equiv\; h_{t\,4.1} \;=\; \frac{1 - f_{\rm o} - f_{\rm c} + f_{\rm f}}{1 - f_{\rm o} + f_{\rm f}}\, h_{t\,4} \;+\; \frac{f_{\rm c}}{1 - f_{\rm o} + f_{\rm f}}\, h_{t\,3} \tag{172}$$

$$T_{\rm guess} \;=\; \frac{1 - f_{\rm o} - f_{\rm c} + f_{\rm f}}{1 - f_{\rm o} + f_{\rm f}}\, T_{t\,4} \;+\; \frac{f_{\rm c}}{1 - f_{\rm o} + f_{\rm f}}\, T_{t\,3} \tag{173}$$

$$T_{t\,4.1} \;=\; \mathcal{F}_T\left(\vec{\lambda}', h_{t\,4.1}\,;\, T_{\rm guess}\right) \tag{174}$$

The total pressure is then obtained using the mixed-out velocity $u_{4.1}$, together with the $\mathcal{F}_h$ function.

$$h_{4.1} \;=\; h_{t\,4.1} \;-\; \tfrac{1}{2}\, u_{4.1}^2 \tag{175}$$

$$\Delta h \;=\; \tfrac{1}{2}\, u_{4.1}^2 \tag{176}$$

$$\{\, p_{t\,4.1}, T_{t\,4.1}, h_{t\,4.1} \,\} \;=\; \mathcal{F}_h\left(\vec{\lambda}', p_{4.1}, T_{4.1}, \Delta h, 1\right) \tag{177}$$

4.8 Turbine quantities

4.8.1 High Pressure Turbine

The HPT enthalpy drop is obtained by equating the windage and friction-loss discounted turbine work with the HPC work.

$$(\dot{m}_{\rm core} - \dot{m}_{\rm offt} + \dot{m}_{\rm fuel})\,(h_{t\,4.1} - h_{t\,4.5})\,(1 - \epsilon_{\rm h}) \;=\; (\dot{m}_{\rm core} - \dot{m}_{\rm offt})\,(h_{t\,3} - h_{t\,2.5}) \tag{178}$$

$$\Delta h_{\rm ht} \;\equiv\; h_{t\,4.5} - h_{t\,4.1} \tag{179}$$

$$\;=\; \frac{-(1 - f_{\rm o})}{1 - f_{\rm o} + f_{\rm f}} \left[\, h_{t\,3} - h_{t\,2.5} \,\right] \frac{1}{1 - \epsilon_{\rm h}} \tag{180}$$

This enthalpy drop, together with an assumed polytropic efficiency, is then used to determine the HPT exit stagnation conditions.

$$\{p_{t\,4.5}, T_{t\,4.5}, h_{t\,4.5}\} \;=\; \mathcal{F}_h\left(\vec{\lambda}', p_{t\,4.1}, T_{t\,4.1}, \Delta h_{\rm ht}, {\eta_{\rm pol_{ht}}}^{-1}\right) \tag{181}$$

4.8.2 Low Pressure Turbine (Design case)

The LPT enthalpy drop for the design case is obtained by equating the windage friction-loss discounted turbine work with the LPC plus fan work, plus any shaft power offtake.

$$
\begin{aligned}
&(\dot{m}_{\mathrm{core}} - \dot{m}_{\mathrm{offt}} + \dot{m}_{\mathrm{fuel}})\,(h_{t\,4.5} - h_{t\,4.9})\,(1-\epsilon_1) \\
&\qquad = \dot{m}_{\mathrm{core}}\,(h_{t\,2.5} - h_{t\,1.9}) \;+\; \dot{m}_{\mathrm{fan}}\,(h_{t\,2.1} - h_{t\,2}) \;+\; P_{\mathrm{offt}} \qquad (182)
\end{aligned}
$$

$$
\Delta h_{\mathrm{lt}} \equiv h_{t\,4.9} - h_{t\,4.5} \qquad (183)
$$

$$
= \frac{-1}{1 - f_{\mathrm{o}} + f_{\mathrm{f}}} \left[(h_{t\,2.5} - h_{t\,1.9}) \;+\; \alpha\,(h_{t\,2.1} - h_{t\,2}) \;+\; \frac{P_{\mathrm{offt}}}{\dot{m}_{\mathrm{core}}} \right] \frac{1}{1-\epsilon_1} \qquad (184)
$$

The enthalpy drop calculated above, together with an assumed turbine polytropic efficiency, is then used to determine all the station 4.9 LPT exit conditions. The turbine nozzle total pressure ratio π_{tn} then give the station 5 nozzle conditions.

$$
\begin{aligned}
\{p_{t\,4.9}, T_{t\,4.9}, h_{t\,4.9}\} &= \mathcal{F}_h\left(\vec{\lambda}', p_{t\,4.5}, T_{t\,4.5}, \Delta h_{\mathrm{lt}}, {\eta_{\mathrm{pol}_{\mathrm{lt}}}}^{-1}\right) && (185) \\
p_{t\,5} &= p_{t\,4.9}\,\pi_{\mathrm{tn}} && (186) \\
T_{t\,5} &= T_{t\,4.9} && (187) \\
h_{t\,5} &= h_{t\,4.9} && (188)
\end{aligned}
$$

4.8.3 Low Pressure Turbine (Off-Design case)

The relations above could be used to determine the $(\,)_{t\,5}$ core exit quantities for the off-design case. However, this opens the possibility of $p_{t\,5}$ falling below the nozzle static pressure $p_5 = p_0$ after any one Newton iteration. A common cause is the fan's enthalpy extraction term $\alpha(h_{t\,2.1} - h_{t\,2})$ in (184) being too large because of a momentarily excessive π_{f} and/or $\bar{m}_{\mathrm{f}}$ values, so that Δh_{lt} is too negative which gives a small $p_{t\,5}$ in calculations (185) and (186).

Regardless of the cause, if $p_{t\,5} < p_5$ is a result then the nozzle velocity u_5 cannot be computed, and the subsequent nozzle mass flow and thrust relations cannot be imposed. This causes failure of the overall Newton iteration process. One solution is to underrelax an "excessive" Newton update so that $p_{t\,5}$ never falls below p_5. However, this is rather impractical since a very long calculation chain is required to reach the $p_{t\,5}$ evaluation operations (185) and (186), so the necessary underrelaxation factor cannot be determined without in effect performing and possibly discarding the calculations for one whole Newton iteration.

The solution taken here is to introduce $p_{t\,5}$ as a Newton variable, so that during the Newton update it can be easily monitored to ensure that it never falls below p_5. It also means that the $(\,)_{t\,5}$ quantities are now computed by the alternative procedure of a specified pressure ratio as used for the compressors.

$$
\begin{aligned}
\pi_{\mathrm{lt}} &= \frac{p_{t\,4.9}}{p_{t\,4.5}} = \frac{1}{\pi_{\mathrm{tn}}}\frac{p_{t\,5}}{p_{t\,4.5}} && (189) \\
\{p_{t\,4.9}, T_{t\,4.9}, h_{t\,4.9}\} &= \mathcal{F}_p\left(\vec{\lambda}', p_{t\,4.5}, T_{t\,4.5}, \pi_{\mathrm{lt}}, {\eta_{\mathrm{pol}_{\mathrm{lt}}}}^{-1}\right) && (190) \\
p_{t\,5} &= p_{t\,4.9}\,\pi_{\mathrm{tn}} && (191) \\
T_{t\,5} &= T_{t\,4.9} && (192) \\
h_{t\,5} &= h_{t\,4.9} && (193)
\end{aligned}
$$

The LPT work relation (184) will now play the role as a Newton-system equation which constrains $p_{t\,5}$. This overall procedure will produce the same final result as if $p_{t\,5}$ was calculated from (184), but its Newton iteration behavior is far more stable and reliable.

4.8.4 Turbine efficiencies

The turbine efficiencies (including cooling-air losses) can also be computed out of interest.

$$\{p_{t\,4.5}, (T_{t\,4.5})_{is}, (h_{t\,4.5})_{is}\} = \mathcal{F}_p\left(\vec{\lambda}', p_{t\,4}, T_{t\,4}, p_{t\,4.5}/p_{t\,4}, 1\right) \tag{194}$$

$$\{p_{t\,4.9}, (T_{t\,4.9})_{is}, (h_{t\,4.9})_{is}\} = \mathcal{F}_p\left(\vec{\lambda}', p_{t\,4.5}, T_{t\,4.5}, p_{t\,5}/p_{t\,4.5}, 1\right) \tag{195}$$

$$\eta_{\rm ht} = \frac{h_{t\,4.5} - h_{t\,4}}{(h_{t\,4.5})_{is} - h_{t\,4}} \tag{196}$$

$$\eta_{\rm lt} = \frac{h_{t\,4.9} - h_{t\,4.5}}{(h_{t\,4.9})_{is} - h_{t\,4.5}} \tag{197}$$

4.9 Fan exhaust quantities

The fan exhaust velocity is computed from the known $(\,)_{t\,8}$ fan plume stagnation conditions, and the requirement of ambient exhaust pressure, $p_8 = p_0$.

$$p_{t\,8} = p_{t\,7} \tag{198}$$

$$T_{t\,8} = T_{t\,7} \tag{199}$$

$$\{p_8, T_8, h_8\} = \mathcal{F}_p\left(\vec{\alpha}, p_{t\,8}, T_{t\,8}, p_0/p_{t\,8}, 1\right) \tag{200}$$

$$u_8 = \sqrt{2\,(h_{t\,8} - h_8)} \tag{201}$$

4.10 Core exhaust quantities

The core exhaust velocity is computed from the known $(\,)_{t\,6}$ core plume conditions and the requirement of ambient exhaust pressure, $p_6 = p_0$.

$$p_{t\,6} = p_{t\,5} \tag{202}$$

$$T_{t\,6} = T_{t\,5} \tag{203}$$

$$\{p_6, T_6, h_6\} = \mathcal{F}_p\left(\vec{\lambda}', p_{t\,6}, T_{t\,6}, p_0/p_{t\,6}, 1\right) \tag{204}$$

$$u_6 = \sqrt{2\,(h_{t\,6} - h_6)} \tag{205}$$

4.11 Offtake air exhaust quantities

The offtake air is assumed to have known $T_{t\,9}$ and $p_{t\,9}$ at its discharge nozzle. These are used to compute the discharge flow's velocity u_9 immediately downstream of the nozzle where it reaches ambient pressure p_0. The usual constant–c_p relations are sufficient here.

$$u_9 = \sqrt{2\,c_p T_{t\,9}\left[1 - (p_0/p_{t\,9})^{(\gamma-1)/\gamma}\right]} \tag{206}$$

4.12 Overall engine quantities

The net aircraft mass outflow, mechanical flow power, and jet dissipation are defined as

$$\dot{m} \equiv \oiint \rho \mathbf{V} \cdot \hat{\mathbf{n}} \; \mathrm{d}\mathcal{S} \tag{207}$$

$$P_{K_{\rm inl}} + P_{K_{\rm exit}} \equiv \oiint \left[p - p_\infty + \tfrac{1}{2}\rho \left(V^2 - V_\infty^2 \right) \right] \mathbf{V} \cdot \hat{\mathbf{n}} \; \mathrm{d}\mathcal{S} \tag{208}$$

$$\Phi_{\rm jet} = \dot{E}_{\rm jet} \equiv \oiint \left[(p - p_\infty)(\mathbf{V} - \mathbf{V}_\infty) \cdot \hat{\mathbf{n}} + \tfrac{1}{2}\rho \left| \mathbf{V} - \mathbf{V}_\infty \right|^2 \mathbf{V} \cdot \hat{\mathbf{n}} \right] \; \mathrm{d}\mathcal{S} \tag{209}$$

$$P_{V_{\rm inl}} \equiv \iiint (p - p_\infty) \nabla \cdot \mathbf{V} \; \mathrm{d}\mathcal{V}_{\rm inl} \tag{210}$$

$$\Phi_{\rm inl} \equiv \iiint (\bar{\bar{\boldsymbol{\tau}}} \cdot \nabla) \cdot \mathbf{V} \; \mathrm{d}\mathcal{V}_{\rm inl} \tag{211}$$

$$\Phi_{\rm cc} \equiv \iiint (\bar{\bar{\boldsymbol{\tau}}} \cdot \nabla) \cdot \mathbf{V} \; \mathrm{d}\mathcal{V}_{\rm cc} \tag{212}$$

where the surface integrals are taken over the propulsor outlet and inlet, and also over the offtake air discharge port, with the unit normal $\hat{\mathbf{n}}$ pointing outward. The volume integrals are taken over the inlet streamtube volume $\mathcal{V}_{\rm inl}$, and the core-cowl surface dissipation integral is taken over the fan exhaust streamtube volume $\mathcal{V}_{\rm cc}$ over the exposed core cowl (if any) downstream of the fan nozzle.

In terms of the 1D engine flow quantities, and the assumption $p_6 = p_8 = p_9 = p_0$, these evaluate to

$$\dot{m} = f_{\rm f} \, \dot{m}_{\rm core} \tag{213}$$

$$P_{K_{\rm exit}} = \tfrac{1}{2} \left[(1 - f_{\rm o} + f_{\rm f}) \left(u_6^2 - u_0^2 \right) + \alpha \left(u_8^2 - u_0^2 \right) + f_{\rm o} \left(u_9^2 - u_0^2 \right) \right] \dot{m}_{\rm core} \tag{214}$$

$$\Phi_{\rm jet} = \tfrac{1}{2} \left[(1 - f_{\rm o} + f_{\rm f}) \left(u_6 - u_0 \right)^2 + \alpha \left(u_8 - u_0 \right)^2 + f_{\rm o} \left(u_9 - u_0 \right)^2 \right] \dot{m}_{\rm core} \tag{215}$$

$$P_{K_{\rm inl}} + P_{V_{\rm inl}} = \Phi_{\rm inl} \tag{216}$$

where we set $V_\infty = u_0$ and $p_\infty = p_0$ to use the present terminology.

The net propulsive power is defined and evaluated as

$$\begin{aligned} P_{\rm prop} &\equiv P_{K_{\rm inl}} + P_{V_{\rm inl}} + P_{K_{\rm exit}} - \Phi_{\rm jet} + \dot{m} V_\infty^2 \\ &= \left[(1 - f_{\rm o} + f_{\rm f}) \, u_6 - u_0 + \alpha \, (u_8 - u_0) + f_{\rm o} \, u_9 \right] \dot{m}_{\rm core} \, u_0 + \Phi_{\rm inl} \end{aligned} \tag{217}$$

where the first term in (217) is recognized as the isolated-engine net thrust power, and $\Phi_{\rm inl}$ acts as an effective added thrust power from the ingested boundary layer. A net effective thrust can then be defined and decomposed into components as follows.

$$F \equiv P_{\rm prop}/u_0 = F_6 + F_8 + F_9 + F_{\rm inl} \tag{218}$$

$$F_6 \equiv \left[(1 - f_{\rm o} + f_{\rm f}) \, u_6 - u_0 \right] \dot{m}_{\rm core} \tag{219}$$

$$F_8 \equiv \alpha \, (u_8 - u_0) \, \dot{m}_{\rm core} \tag{220}$$

$$F_9 \equiv f_{\rm o} \, u_9 \, \dot{m}_{\rm core} \tag{221}$$

$$F_{\rm inl} \equiv \Phi_{\rm inl}/u_0 \tag{222}$$

In terms of the inlet kinetic energy defect the dissipation is

$$\Phi_{\rm inl} \simeq \frac{1 + \frac{\gamma - 1}{2} M_{\rm inl}^2}{(1 + \frac{\gamma - 1}{2} M^2)_{\rm avg\,surf}} \, \mathcal{K}_{\rm inl} \tag{223}$$

which is the same as $\mathcal{K}_{\rm inl}$ in low speed flow.

The overall specific power and specific thrust, defined here as

$$P_{\rm sp} \;\equiv\; \frac{P_{\rm prop}}{(1+\alpha)\,\dot{m}_{\rm core}\,u_0^2} \tag{224}$$

$$F_{\rm sp} \;\equiv\; \frac{F}{(1+\alpha)\,\dot{m}_{\rm core}\,u_0} \tag{225}$$

are seen to be equal, and evaluate to

$$P_{\rm sp} \;=\; F_{\rm sp} \;=\; \frac{(1-f_{\rm o}+f_{\rm f})\,u_6 - u_0 \;+\; \alpha(u_8-u_0) \;+\; f_{\rm o}\,u_9}{(1+\alpha)\,u_0} \;+\; \frac{\Phi_{\rm inl}}{(1+\alpha)\,\dot{m}_{\rm core}\,u_0^2} \tag{226}$$

while Kerrebrock [1] defines an alternative specific thrust using the speed of sound.

$$F'_{\rm sp} \;\equiv\; \frac{F}{(1+\alpha)\,\dot{m}_{\rm core}\,a_0} \;=\; M_0\,F_{\rm sp} \tag{227}$$

The overall specific impulse and thrust specific fuel consumption then also follow.

$$I_{\rm sp} \;\equiv\; \frac{F}{\dot{m}_{\rm fuel}\,g} \;=\; \frac{F_{\rm sp}}{f_{\rm f}}\,\frac{u_0}{g}(1+\alpha) \;=\; \frac{F'_{\rm sp}}{f_{\rm f}}\,\frac{a_0}{g}(1+\alpha) \tag{228}$$

$$TSFC \;=\; \frac{1}{I_{\rm sp}} \tag{229}$$

5 Design Sizing Calculation

For the design case, the following quantities are specified in addition to those in Table 1.

$P_{\rm D}$	design net propulsive power
$M_{\rm 2D}$	fan-face, LPC-face axial Mach number
$M_{\rm 2.5D}$	HPC-face axial Mach number

For a BLI configuration, the design net propulsive power is equal to the overall dissipation of the airframe (exludes jet dissipation), plus the added power due any specified climb rate $\dot{h}$ and aircraft weight W. These can also define an equivalent design net thrust.

$$P_{\rm D} \;=\; \Phi_{\rm surf} \;+\; \Phi_{\rm wake} \;+\; \Phi_{\rm vortex} \;+\; W\dot{h} \tag{230}$$

$$F_{\rm D} \;\equiv\; P_{\rm D}/u_0 \tag{231}$$

5.1 Mass Flow Sizing

This consists of finding the core mass flow to achieve the required net propulsive power (or equivalent net thrust) at the design operating conditions p_0, a_0, M_0. This design core mass flow $\dot{m}_{\rm coreD}$ is obtained directly from relation (226), using the specified design equivalent thrust $F_{\rm D}$.

$$\dot{m}_{\rm coreD} \;=\; \frac{F_{\rm D} - F_{\rm inl}}{(1-f_{\rm o}+f_{\rm f})\,u_6 - u_0 \;+\; \alpha(u_8-u_0) \;+\; f_{\rm o}\,u_9} \tag{232}$$

5.2 Component Area Sizing

5.2.1 Inlet areas

The fan-face static ρ_2 and a_2 are obtained from the specified-Mach procedure, with some specified design fan-face Mach number $M_{2\mathrm{D}}$, with a total pressure defined in terms of the inlet recovery ratio π_d and any BLI entropy rise $\bar{s}$ defined by (126).

$$p_{t\,2} \;=\; p_{t\,1.8}\,\exp(-\bar{s}) \;=\; p_{t\,0}\,\pi_\mathrm{d}\,\exp(-\bar{s}) \tag{233}$$
$$T_{t\,2} \;=\; T_{t\,0} \tag{234}$$
$$\{p_2, T_2, h_2, c_{p_2}, R_2\} \;=\; \mathcal{F}_M\left(\vec{\alpha}, p_{t\,2}, T_{t\,2}, 0, M_{2\mathrm{D}}, 1\right) \tag{235}$$

The same calculation is performed for the LPC inlet state.

$$\{p_{1.9}, T_{1.9}, h_{1.9}, c_{p_{1.9}}, R_{1.9}\} \;=\; \mathcal{F}_M\left(\vec{\alpha}, p_{t\,1.9}, T_{t\,1.9}, 0, M_{2\mathrm{D}}, 1\right) \tag{236}$$

The overall fan+LPC area A_2 is then computed from the design bypass and core mass flows.

$$\rho_2 \;=\; \frac{p_2}{R_2 T_2} \tag{237}$$
$$u_2 \;=\; \sqrt{2\,(h_{t\,2} - h_2)} \tag{238}$$
$$\rho_{1.9} \;=\; \frac{p_{1.9}}{R_{1.9} T_{1.9}} \tag{239}$$
$$u_{1.9} \;=\; \sqrt{2\,(h_{t\,1.9} - h_{1.9})} \tag{240}$$
$$A_2 \;=\; \alpha\,\frac{\dot{m}_\mathrm{coreD}}{\rho_2\, u_2} \;+\; \frac{\dot{m}_\mathrm{coreD}}{\rho_{1.9}\, u_{1.9}} \tag{241}$$

A specified hub/tip ratio HTR_f then also gives the fan diameter,

$$d_\mathrm{f} \;=\; \sqrt{\frac{4}{\pi}\frac{A_2}{1 - HTR_\mathrm{f}^2}} \tag{242}$$

although this is not required for any subsequent off-design analysis.

The HP compressor fan area $A_{2.5}$ is obtained in this same manner from a specified compressor-face Mach number $M_{2.5\mathrm{D}}$.

$$\{p_{2.5}, T_{2.5}, c_{p_{2.5}}, R_{2.5}\} \;=\; \mathcal{F}_M\left(\vec{\alpha}, p_{t\,2.5}, T_{t\,2.5}, 0, M_{2.5\mathrm{D}}, 1\right) \tag{243}$$
$$\rho_{2.5} \;=\; \frac{p_{2.5}}{R_{2.5} T_{2.5}} \tag{244}$$
$$u_{2.5} \;=\; M_{2.5\mathrm{D}} \sqrt{\frac{c_{p_{2.5}}\, R_{2.5}}{c_{p_{2.5}} - R_{2.5}}\, T_{2.5}} \tag{245}$$
$$A_{2.5} \;=\; (1 - f_\mathrm{o})\,\frac{\dot{m}_\mathrm{coreD}}{\rho_{2.5}\, u_{2.5}} \tag{246}$$

A specified hub/tip ratio HTR_hc then also gives the HPC face diameter.

$$d_\mathrm{hc} \;=\; \sqrt{\frac{4}{\pi}\frac{A_{2.5}}{1 - HTR_\mathrm{hc}^2}} \tag{247}$$

5.2.2 Fan nozzle area

The fan nozzle flow type can be determined from the fan-plume Mach number.

$$M_8 \;=\; u_8 \Big/ \sqrt{\frac{c_{p_8}\, R_8}{c_{p_8} - R_8}\, T_8} \tag{248}$$

If $M_8 < 1$ then the fan nozzle is assumed to be unchoked, and the nozzle conditions are obtained by using the specified pressure ratio function. The nozzle is assumed here to be at ambient static pressure, although any other pressure can be specified instead.

$$p_7 \;=\; p_0 \tag{249}$$

$$\{p_7, T_7, h_7\} \;=\; \mathcal{F}_p\left(\vec{\alpha}, p_{t\,7}, T_{t\,7}, p_7/p_{t\,7}, 1\right) \tag{250}$$

If $M_8 \geq 1$ then the fan nozzle is choked, and the nozzle conditions are obtained using the specified Mach function.

$$M_7 \;=\; 1 \tag{251}$$

$$\{p_7, T_7, h_7\} \;=\; \mathcal{F}_M\left(\vec{\alpha}, p_{t\,7}, T_{t\,7}, 0, M_7, 1\right) \tag{252}$$

In either case, the fan nozzle area follows directly.

$$u_7 \;=\; \sqrt{2\left(h_{t\,7} - h_7\right)} \tag{253}$$

$$\rho_7 \;=\; \frac{p_7}{R_7 T_7} \tag{254}$$

$$A_7 \;=\; \alpha\, \frac{\dot{m}_{\mathrm{coreD}}}{\rho_7\, u_7} \tag{255}$$

5.2.3 Core nozzle area

The core nozzle flow type is determined from the core-plume Mach number.

$$M_6 \;=\; u_6 \Big/ \sqrt{\frac{c_{p_6}\, R_6}{c_{p_6} - R_6}\, T_6} \tag{256}$$

If $M_6 < 1$ then the core nozzle is unchoked, and the nozzle conditions are obtained by using the specified pressure ratio function.

$$p_5 \;=\; p_0 \tag{257}$$

$$\{p_5, T_5, h_5\} \;=\; \mathcal{F}_p\left(\vec{\alpha}, p_{t\,5}, T_{t\,5}, p_5/p_{t\,5}, 1\right) \tag{258}$$

If $M_6 \geq 1$ then the fan nozzle is choked, and the nozzle conditions are obtained using the specified Mach function.

$$M_5 \;=\; 1 \tag{259}$$

$$\{p_5, T_5, h_5\} \;=\; \mathcal{F}_M\left(\vec{\alpha}, p_{t\,5}, T_{t\,5}, 0, M_5, 1\right) \tag{260}$$

The core nozzle area follows.

$$u_5 \;=\; \sqrt{2\left(h_{t\,5} - h_5\right)} \tag{261}$$

$$\rho_5 \;=\; \frac{p_5}{R_5 T_5} \tag{262}$$

$$A_5 \;=\; (1 - f_{\mathrm{o}} + f_{\mathrm{f}})\, \frac{\dot{m}_{\mathrm{coreD}}}{\rho_5\, u_5} \tag{263}$$

5.2.4 Offtake air nozzle area

The offtake air density can be computed from the usual constant–c_p relations. The offtake air discharge nozzle area then follows.

$$p_9 = p_0 \tag{264}$$

$$T_9 = T_{t\,9}\,(p_9/p_{t\,9})^{(\gamma-1)/\gamma} \tag{265}$$

$$\rho_9 = \frac{p_9}{R_0 T_9} \tag{266}$$

$$A_9 = f_{\rm o}\,\frac{\dot{m}_{\rm coreD}}{\rho_9\,u_9} \tag{267}$$

5.3 Design corrected speeds and mass flows

Since only speed ratios will be considered in the off-design calculation, the LPC and HPC design spool speeds can be arbitrarily set to unity.

$$N_{\rm lD} = 1 \tag{268}$$

$$N_{\rm hD} = 1 \tag{269}$$

The design corrected spool speeds and design corrected mass flows are defined in the usual manner.

$$\bar{N}_{\rm lD} = N_{\rm lD}\,\frac{1}{\sqrt{T_{t\,1.9}/T_{\rm ref}}} \tag{270}$$

$$\bar{N}_{\rm hD} = N_{\rm hD}\,\frac{1}{\sqrt{T_{t\,2.5}/T_{\rm ref}}} \tag{271}$$

$$\bar{m}_{\rm f\,D} = \alpha\,\dot{m}_{\rm coreD}\,\frac{\sqrt{T_{t\,2}/T_{\rm ref}}}{p_{t\,2}/p_{\rm ref}} \tag{272}$$

$$\bar{m}_{\rm lcD} = \dot{m}_{\rm coreD}\,\frac{\sqrt{T_{t\,1.9}/T_{\rm ref}}}{p_{t\,1.9}/p_{\rm ref}} \tag{273}$$

$$\bar{m}_{\rm hcD} = (1-f_{\rm o})\,\dot{m}_{\rm coreD}\,\frac{\sqrt{T_{t\,2.5}/T_{\rm ref}}}{p_{t\,2.5}/p_{\rm ref}} \tag{274}$$

$$\bar{m}_{\rm htD} = (1-f_{\rm o}+f_{\rm f})\,\dot{m}_{\rm coreD}\,\frac{\sqrt{T_{t\,4.1}/T_{\rm ref}}}{p_{t\,4.1}/p_{\rm ref}} \tag{275}$$

$$\bar{m}_{\rm ltD} = (1-f_{\rm o}+f_{\rm f})\,\dot{m}_{\rm coreD}\,\frac{\sqrt{T_{t\,4.5}/T_{\rm ref}}}{p_{t\,4.5}/p_{\rm ref}} \tag{276}$$

6 Off-Design Operation Calculation

For an off-design case, the following eight quantities which were assumed known at the start of the calculation pass are really unknowns, and must be updated.

$\pi_{\rm f}$	fan pressure ratio	
$\pi_{\rm lc}$	LPC pressure ratio	
$\pi_{\rm hc}$	HPC pressure ratio	
$\bar{m}_{\rm f}$	fan corrected mass flow	$\equiv \alpha\, \dot{m}_{\rm core} \sqrt{T_{t\,2}\,/T_{\rm ref}}\;\; p_{\rm ref}/p_{t\,2}$
$\bar{m}_{\rm lc}$	LPC corrected mass flow	$\equiv \dot{m}_{\rm core} \sqrt{T_{t\,1.9}/T_{\rm ref}}\;\; p_{\rm ref}/p_{t\,1.9}$
$\bar{m}_{\rm hc}$	HPC corrected mass flow	$\equiv (1-f_{\rm o})\dot{m}_{\rm core} \sqrt{T_{t\,2.5}/T_{\rm ref}}\;\; p_{\rm ref}/p_{t\,2.5}$
$T_{t\,4}$	combustor exit total temperature	
$p_{t\,5}$	core nozzle total pressure	

The necessary eight constraining equations involve the spool speeds, which are calculated as described in the next section. The speed calculation is based on an assumed fan or compressor map, and has the following functional form.

$$\bar{N} \;=\; \mathcal{F}_N(\pi\,,\, \bar{m}\,;\, \pi_{\rm D}\,,\, \bar{m}_{\rm D}) \tag{277}$$

This is used to compute the fan, LPC, and HPC speed from each component's current pressure ratio and corrected mass flows. The current station 1.9, 2, 2.5 stagnation conditions are also used, to compute the necessary $\bar{m}$ arguments for the $\mathcal{F}_N$ functions.

$$N_{\rm f} \;=\; \sqrt{T_{t\,2}/T_{\rm ref}}\;\; \mathcal{F}_N(\pi_{\rm f}\,,\, \bar{m}_{\rm f}\,;\, \pi_{\rm f\,D}\,,\, \bar{m}_{\rm f\,D}) \tag{278}$$

$$N_{\rm l} \;=\; \sqrt{T_{t\,1.9}/T_{\rm ref}}\;\; \mathcal{F}_N(\pi_{\rm lc}\,,\, \bar{m}_{\rm lc}\,;\, \pi_{\rm lcD}\,,\, \bar{m}_{\rm lcD}) \tag{279}$$

$$N_{\rm h} \;=\; \sqrt{T_{t\,2.5}/T_{\rm ref}}\;\; \mathcal{F}_N(\pi_{\rm hc}\,,\, \bar{m}_{\rm hc}\,;\, \pi_{\rm hcD}\,,\, \bar{m}_{\rm hcD}) \tag{280}$$

Each of these three functions uses the appropriate map constants for that component, given in the component-map section.

The off-design fan face Mach number M_2 can be calculated from the fan-face mass flow relation.

$$\rho_2 u_2 A_2 \;=\; \bar{m}_{\rm f} \sqrt{\frac{T_{\rm ref}}{T_{t\,2}}}\, \frac{p_{t\,2}}{p_{\rm ref}} \;+\; \bar{m}_{\rm lc} \sqrt{\frac{T_{\rm ref}}{T_{t\,1.9}}}\, \frac{p_{t\,1.9}}{p_{\rm ref}} \;=\; (1+\alpha)\, \dot{m}_{\rm core} \tag{281}$$

This is solved for the implied M_2 using the station 2 specified mass/area m' and stagnation quantities in the $\mathcal{F}_m$ function. The design M_2 value is a suitable initial guess, and also specifies the subsonic branch.

$$M_{\rm guess} \;=\; M_{2{\rm D}} \tag{282}$$

$$m' \;=\; (1+\alpha)\, \dot{m}_{\rm core}/A_2 \tag{283}$$

$$\{p_2, T_2, h_2\} \;=\; \mathcal{F}_m\,(\vec{\alpha}, p_{t\,2}, T_{t\,2}, h_{t\,2}, m'\,;\, M_{\rm guess}) \tag{284}$$

$$u_2 \;=\; \sqrt{2\,(h_{t\,2} - h_2)} \tag{285}$$

$$M_2 \;=\; u_2/\sqrt{\frac{c_{p_2}\, R_2}{c_{p_2} - R_2}\, T_2} \tag{286}$$

6.1 Constraint residuals

The eight residuals for constraining the eight operating unknowns are listed below.

6.1.1 Fan/LPC speed constraint

Equating the fan and LPC speeds, with some specified gear ratio G_f, defines the constraining equation residual.

$$\mathcal{R}_1 \;\equiv\; N_\mathrm{f}\, G_\mathrm{f} \;-\; N_\mathrm{l} \;=\; 0 \tag{287}$$

6.1.2 HPT mass flow

In lieu of a full turbine map, it is reasonable to assume that the high-pressure turbine IGV is always choked. The appropriate constraining residual is therefore a fixed corrected mass flow at station 4.1 (conveniently defined in terms of the LPC corrected mass flow variable $\bar{m}_\mathrm{lc}$ and the offtake and fuel fractions), set equal to the design value.

$$\bar{m}_\mathrm{ht} \;=\; (1-f_\mathrm{o}+f_\mathrm{f})\,\bar{m}_\mathrm{lc}\sqrt{\frac{T_{t\,4.1}}{T_{t\,1.9}}}\,\frac{p_{t\,1.9}}{p_{t\,4.1}} \tag{288}$$

$$\mathcal{R}_2 \;\equiv\; \bar{m}_\mathrm{ht} \;-\; \bar{m}_\mathrm{htD} \;=\; 0 \tag{289}$$

Note that this approximation means that the high spool speed N_h is not required in any of the calculations.

6.1.3 LPT mass flow

The low-pressure turbine IGV is also assumed to be choked. Again, the constraining residual is a fixed corrected mass flow at station 4.5, equal to the design value.

$$\bar{m}_\mathrm{lt} \;=\; (1-f_\mathrm{o}+f_\mathrm{f})\,\bar{m}_\mathrm{lc}\sqrt{\frac{T_{t\,4.5}}{T_{t\,1.9}}}\,\frac{p_{t\,1.9}}{p_{t\,4.5}} \tag{290}$$

$$\mathcal{R}_3 \;\equiv\; \bar{m}_\mathrm{lt} \;-\; \bar{m}_\mathrm{ltD} \;=\; 0 \tag{291}$$

6.1.4 Fan nozzle mass flow

The type of constraint imposed at the fan nozzle depends on whether or not the nozzle is choked. The fan nozzle trial static conditions and trial Mach number $\tilde{M}_7$ are first computed assuming a specified nozzle static pressure, equal to the freestream pressure.

$$\tilde{p}_7 \;=\; p_0 \tag{292}$$

$$\left\{\tilde{p}_7, \tilde{T}_7, \tilde{h}_7\right\} \;=\; \mathcal{F}_p\left(\vec{\alpha}, p_{t\,7}, T_{t\,7}, p_7/p_{t\,7}, 1\right) \tag{293}$$

$$\tilde{u}_7 \;=\; \sqrt{2\left(h_{t\,7} - \tilde{h}_7\right)} \tag{294}$$

$$\tilde{M}_7 \;=\; \tilde{u}_7 \Big/ \sqrt{\frac{\tilde{c}_{p_7}\, R_7}{\tilde{c}_{p_7} - R_7}\,\tilde{T}_7} \tag{295}$$

If $\tilde{M}_7 \leq 1$, then the trial state is the actual state.

$$p_7 \;=\; \tilde{p}_7 \tag{296}$$

$$T_7 \;=\; \tilde{T}_7 \tag{297}$$

$$h_7 \;=\; \tilde{h}_7 \tag{298}$$

If $\tilde{M}_7 > 1$, then the trial state is incorrect, and a unity Mach number is imposed instead.

$$M_7 = 1 \quad (299)$$
$$\{p_7, T_7, h_7\} = \mathcal{F}_M(\vec{\alpha}, p_{t\,7}, T_{t\,7}, 0, M_7, 1) \quad (300)$$

For either case, the velocity, density, and mass flow constraint residual is formulated the same way.

$$u_7 = \sqrt{2(h_{t\,7} - h_7)} \quad (301)$$
$$\rho_7 = \frac{p_7}{R_7 T_7} \quad (302)$$
$$\mathcal{R}_4 \equiv \bar{m}_{\rm f} \sqrt{\frac{T_{\rm ref}}{T_{t\,2}}} \frac{p_{t\,2}}{p_{\rm ref}} - \rho_7 u_7 A_7 = 0 \quad (303)$$

6.1.5 Core nozzle mass flow

The type of constraint imposed at the core nozzle depends on whether or not the nozzle is choked. The core nozzle trial static conditions and trial Mach number $\tilde{M}_5$ are first computed assuming a specified nozzle static pressure, equal to the freestream pressure.

$$\tilde{p}_5 = p_0 \quad (304)$$
$$\left\{\tilde{p}_5, \tilde{T}_5, \tilde{h}_5\right\} = \mathcal{F}_p(\vec{\alpha}, p_{t\,5}, T_{t\,5}, p_5/p_{t\,5}, 1) \quad (305)$$
$$\tilde{u}_5 = \sqrt{2(h_{t\,5} - \tilde{h}_5)} \quad (306)$$
$$\tilde{M}_5 = \tilde{u}_5 / \sqrt{\frac{\tilde{c}_{p_5} R_5}{\tilde{c}_{p_5} - R_5} \tilde{T}_5} \quad (307)$$

If $\tilde{M}_5 \leq 1$, then the trial state is the actual state.

$$p_5 = \tilde{p}_5 \quad (308)$$
$$T_5 = \tilde{T}_5 \quad (309)$$
$$h_5 = \tilde{h}_5 \quad (310)$$

If $\tilde{M}_5 > 1$, then the trial state is incorrect, and a unity Mach number is imposed instead.

$$M_5 = 1 \quad (311)$$
$$\{p_5, T_5, h_5\} = \mathcal{F}_M(\vec{\alpha}, p_{t\,5}, T_{t\,5}, 0, M_5, 1) \quad (312)$$

For either case, the velocity, density, and mass flow constraint residual is formulated the same way.

$$u_5 = \sqrt{2(h_{t\,5} - h_5)} \quad (313)$$
$$\rho_5 = \frac{p_5}{R_5 T_5} \quad (314)$$
$$\mathcal{R}_5 \equiv (1 - f_{\rm o} + f_{\rm f})\, \bar{m}_{\rm lc} \sqrt{\frac{T_{\rm ref}}{T_{t\,1.9}}} \frac{p_{t\,1.9}}{p_{\rm ref}} - \rho_5 u_5 A_5 = 0 \quad (315)$$

6.1.6 LPC/HPC mass flow constraint

Equating the LPC and HPC mass flows, with allowance for any bleed mass flow offtake, defines the sixth constraining equation residual.

$$\mathcal{R}_6 \;\equiv\; \bar{m}_{\rm lc}\sqrt{\frac{T_{\rm ref}}{T_{t\,1.9}}}\,\frac{p_{t\,1.9}}{p_{\rm ref}} \;-\; \bar{m}_{\rm hc}\sqrt{\frac{T_{\rm ref}}{T_{t\,2.5}}}\,\frac{p_{t\,2.5}}{p_{\rm ref}} \;-\; \dot{m}_{\rm offt} \;=\; 0 \qquad (316)$$

6.1.7 Combustor exit temperature constraint

One of two possible constraints on $T_{t\,4}$ can be used.

$$\mathcal{R}_7 \;\equiv\; T_{t\,4} \;-\; (T_{t\,4})_{\rm spec} \;=\; 0 \qquad (T_{t\,4} \text{ specified}) \qquad (317)$$

$$\mathcal{R}_7 \;\equiv\; F \;-\; F_{\rm spec} \;=\; 0 \qquad (\text{thrust specified}) \qquad (318)$$

The thrust F is defined by relation (218), and is ultimately a function of the eight Newton variables.

6.1.8 Core exit total pressure constraint

The constraint on $p_{t\,5}$ is obtained from the LPT work relations (184) and (185), which have not been used yet for the off-design case.

$$\frac{\dot{m}_{\rm offt}}{\dot{m}_{\rm core}} \;\equiv\; f_{\rm o} \;=\; \frac{\dot{m}_{\rm offt}}{\bar{m}_{\rm lc}}\sqrt{\frac{T_{t\,1.9}}{T_{\rm ref}}}\,\frac{p_{\rm ref}}{p_{t\,1.9}} \qquad (319)$$

$$\frac{P_{\rm offt}}{\dot{m}_{\rm core}} \;\equiv\; \mathcal{P}_{\rm o} \;=\; \frac{P_{\rm offt}}{\bar{m}_{\rm lc}}\sqrt{\frac{T_{t\,1.9}}{T_{\rm ref}}}\,\frac{p_{\rm ref}}{p_{t\,1.9}} \qquad (320)$$

$$\frac{\dot{m}_{\rm fan}}{\dot{m}_{\rm core}} \;\equiv\; \alpha \;=\; \frac{\bar{m}_{\rm f}}{\bar{m}_{\rm lc}}\sqrt{\frac{T_{t\,1.9}}{T_{t\,2}}}\,\frac{p_{t\,2}}{p_{t\,1.9}} \qquad (321)$$

$$\Delta h_{\rm lt} \;=\; \frac{-1}{1-f_{\rm o}+f_{\rm f}}\left[(h_{t\,2.5}-h_{t\,1.9}) \;+\; \alpha\,(h_{t\,2.1}-h_{t\,2}) \;+\; \mathcal{P}_{\rm o}\right]\frac{1}{1-\epsilon_1} \qquad (322)$$

$$p_{t\,4.9} \;=\; \mathcal{F}_h\left(\vec{\lambda}', p_{t\,4.5}, T_{t\,4.5}, \Delta h_{\rm lt}, {\eta_{\rm pol}}_{\rm lt}^{-1}\right) \qquad (323)$$

$$\mathcal{R}_8 \;\equiv\; p_{t\,5} \;-\; p_{t\,4.9}\,\pi_{\rm tn} \;=\; 0 \qquad (324)$$

6.1.9 Inlet Mach number constraint

The off-design inlet Mach number is implicitly constrained by the fan-face mass flow relation (241) used to size the fan area, but with A_2 now held fixed.

$$\{p_2, T_2, h_2\} \;=\; \mathcal{F}_M\,(\vec{\alpha}, p_{t\,2}, T_{t\,2}, 0, M_2, 1) \qquad (325)$$

$$\rho_2 \;=\; \frac{p_2}{R_2 T_2} \qquad (326)$$

$$u_2 \;=\; \sqrt{2\,(h_{t\,2}-h_2)} \qquad (327)$$

$$\{p_{1.9}, T_{1.9}, h_{1.9}\} \;=\; \mathcal{F}_M\,(\vec{\alpha}, p_{t\,1.9}, T_{t\,1.9}, 0, M_2, 1) \qquad (328)$$

$$\rho_{1.9} \;=\; \frac{p_{1.9}}{R_{1.9} T_{1.9}} \qquad (329)$$

$$u_{1.9} \;=\; \sqrt{2\,(h_{t\,1.9}-h_{1.9})} \qquad (330)$$

$$\mathcal{R}_9 \;\equiv\; \bar{m}_{\rm f}\sqrt{\frac{T_{\rm ref}}{T_{t\,2}}}\,\frac{p_{t\,2}}{p_{\rm ref}}\,\frac{1}{\rho_2 u_2} \;+\; \bar{m}_{\rm lc}\sqrt{\frac{T_{\rm ref}}{T_{t\,1.9}}}\,\frac{p_{t\,1.9}}{p_{\rm ref}}\,\frac{1}{\rho_{1.9} u_{1.9}} \;-\; A_2 \;=\; 0 \quad (331)$$

6.2 Newton update

The nine residuals depend explicitly or implicitly on the nine unknowns. Newton changes are computed by forming and solving the 9×9 linear Newton system.

$$\left[\frac{\partial(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4,\mathcal{R}_5,\mathcal{R}_6,\mathcal{R}_7,\mathcal{R}_8,\mathcal{R}_9)}{\partial(\pi_{\rm f},\pi_{\rm lc},\pi_{\rm hc},\bar{m}_{\rm f},\bar{m}_{\rm lc},\bar{m}_{\rm hc},T_{t\,4},p_{t\,5},M_2)}\right]\left\{\begin{array}{c}\delta\pi_{\rm f}\\ \delta\pi_{\rm lc}\\ \delta\pi_{\rm hc}\\ \delta\bar{m}_{\rm f}\\ \delta\bar{m}_{\rm lc}\\ \delta\bar{m}_{\rm hc}\\ \delta T_{t\,4}\\ \delta p_{t\,5}\\ \delta M_2\end{array}\right\} = -\left\{\begin{array}{c}\mathcal{R}_1\\ \mathcal{R}_2\\ \mathcal{R}_3\\ \mathcal{R}_4\\ \mathcal{R}_5\\ \mathcal{R}_6\\ \mathcal{R}_7\\ \mathcal{R}_8\\ \mathcal{R}_9\end{array}\right\} \quad (332)$$

The Newton changes are then used to update the variables,

$$\left\{\begin{array}{c}\pi_{\rm f}\\ \pi_{\rm lc}\\ \pi_{\rm hc}\\ \bar{m}_{\rm f}\\ \bar{m}_{\rm lc}\\ \bar{m}_{\rm hc}\\ T_{t\,4}\\ p_{t\,5}\\ M_2\end{array}\right\} \leftarrow \left\{\begin{array}{c}\pi_{\rm f}\\ \pi_{\rm lc}\\ \pi_{\rm hc}\\ \bar{m}_{\rm f}\\ \bar{m}_{\rm lc}\\ \bar{m}_{\rm hc}\\ T_{t\,4}\\ p_{t\,5}\\ M_2\end{array}\right\} + \;\omega\left\{\begin{array}{c}\delta\pi_{\rm f}\\ \delta\pi_{\rm lc}\\ \delta\pi_{\rm hc}\\ \delta\bar{m}_{\rm f}\\ \delta\bar{m}_{\rm lc}\\ \delta\bar{m}_{\rm hc}\\ \delta T_{t\,4}\\ \delta p_{t\,5}\\ \delta M_2\end{array}\right\} \quad (333)$$

where $\omega \leq 1$ is possible underrelaxation factor set so that the resulting new variables stay within physically-dictated limits, e.g. $\pi > 1$, $p_{t\,5} > p_0$, $M_2 < 1$, etc. This is usually required only for the first few iterations where the Newton changes are large. Once the solution is approached and the changes become small, $\omega = 1$ is used.

After the update, all calculations are repeated for the next Newton iteration. Typically, 4–10 iterations are required for convergence to machine zero.

7 Fan and Compressor Maps

To enforce the fan/compressor speed matching requirement, the fan and compressor speeds are determined from their pressure ratios and mass flows. Also, it is desirable to obtain realistic degraded efficiencies away from the design point. These are implemented here using approximate canonincal compressor pressure-ratio maps and efficiency maps.

7.1 Pressure ratio map

The corrected speed and mass flow is defined in the usual way,

$$\bar{N} \;=\; N \frac{1}{\sqrt{T_{ti}/T_{\rm ref}}} \tag{334}$$

$$\bar{m} \;=\; \dot{m} \frac{\sqrt{T_{ti}/T_{\rm ref}}}{p_{ti}/p_{\rm ref}} \tag{335}$$

where T_{ti}, p_{ti} are the face quantities, either $T_{t\,2}, p_{t\,2}$ for the fan and the LPC, or $T_{t\,2.5}, p_{t\,2.5}$ for the HPC.

The fan and compressor maps are in turn defined in terms of these corrected values normalized by their design values.

$$\tilde{p} \;=\; \frac{\pi - 1}{\pi_{\rm D} - 1} \tag{336}$$

$$\tilde{m} \;=\; \frac{\bar{m}}{\bar{m}_{\rm D}} \tag{337}$$

$$\tilde{N} \;=\; \frac{\bar{N}}{\bar{N}_{\rm D}} \tag{338}$$

The "spine" $\tilde{p}_s(\tilde{m}_s)$ on which the speed lines are threaded is parameterized by the corrected speed in the form

$$\tilde{m}_{s(\tilde{N})} \;=\; \tilde{N}^b \tag{339}$$

$$\tilde{p}_{s(\tilde{N})} \;=\; \tilde{m}_s^a \;=\; \tilde{N}^{ab} \tag{340}$$

where a controls the shape of the spine, and b controls the positioning of the speed lines along the spine.

The "knee" shape of each speed line is assumed to be a simple logarithmic function, translated to the $\tilde{m}_s, \tilde{p}_s$ position along the spine.

$$\tilde{p} - \tilde{p}_s \;=\; 2\tilde{N}\, k\; \ln\!\left(1 - \frac{\tilde{m} - \tilde{m}_s}{k}\right) \tag{341}$$

The constant k controls the sharpness of the logarithmic knee. Function (341) can be recast into an explicit form of a usual compressor map.

$$\pi\left(\bar{m}, \bar{N}\right) \;=\; 1 \;+\; (\pi_{\rm D} - 1)\left[\tilde{N}^{ab} \;+\; 2\tilde{N}\, k\; \ln\!\left(1 - \frac{\tilde{m} - \tilde{N}^b}{k}\right)\right] \tag{342}$$

Equation (342) is actually used here in inverse form, giving the fan or compressor corrected speed as a function of the pressure ratio and corrected mass flow.

$$\bar{N} \;=\; \mathcal{F}_N\left(\pi, \pi_{\rm D}, \bar{m}, \bar{m}_{\rm D}\right) \tag{343}$$

This is implemented by inverting the map (341) using the Newton method. To avoid problems with the extremely nonlinear logarithmic shape of each speed line curve, the Newton residual of (342) is formulated in one of two equivalent ways, depending on whether the specified $\tilde{m}, \tilde{p}$ point is above or below the spine curve (see Figure 3).

$$\mathcal{R}(\tilde{N}) \;=\; \tilde{p}_s(\tilde{N}) \;+\; 2\tilde{N}\,k\;\ln\!\left(1 - \frac{\tilde{m} - \tilde{m}_s(\tilde{N})}{k}\right) \;-\; \tilde{p} \qquad (\text{if } \tilde{p} \geq \tilde{m}^a) \tag{344}$$

$$\mathcal{R}(\tilde{N}) \;=\; \tilde{m}_s(\tilde{N}) \;+\; k\left[1 \;-\; \exp\left(\frac{\tilde{p} - \tilde{p}_s(\tilde{N})}{2\tilde{N}k}\right)\right] \;-\; \tilde{m} \qquad (\text{if } \tilde{p} < \tilde{m}^a) \tag{345}$$

Residual (344), used above the spine curve, drives to the intersection of a speed line curve with a vertical constant-$\tilde{m}$ line. Residual (345), used below the spine curve, drives to the intersection of a speed line curve with a horizontal constant-$\tilde{p}$ line. In each case the intersection is nearly orthogonal, giving an extremely stable Newton iteration with rapid convergence in all cases.

7.2 Polytropic efficiency

The polytropic efficiency function is assumed to have the following form.

$$\eta_{\text{pol}} \;=\; \mathcal{F}_\eta(\pi, \pi_{\text{D}}, \bar{m}, \bar{m}_{\text{D}}) \;\equiv\; \eta_{\text{pol}_o}\left[1 \;-\; C\left|\frac{\tilde{p}}{\tilde{m}^{a+\Delta a-1}} \;-\; \tilde{m}\right|^c \;-\; D\left|\frac{\tilde{m}}{\tilde{m}_o} \;-\; 1\right|^d\right] \tag{346}$$

The maximum efficiency is η_{pol_o}, located at $\tilde{m} = \tilde{m}_o$ and $\tilde{p} = \tilde{m}_o^{(a+\Delta a)}$ along the spine of the efficiency map. The exponent of the spine is $a + \Delta a$, which differs from the exponent of the pressure-map spine by the small amount Δa. Typically, Δa is slightly negative for single-stage fans, and slightly positive for multi-stage compressors. The c, d, C, D constants control the decrease of η_{pol} as $\tilde{m}, \tilde{p}$ move away from the $\tilde{m}_o, \tilde{p}_o$ point.

7.3 Map calibration

The constants in Table 2 give a realistic fan map, which is compared to the E^3 fan data. The resulting pressure-ratio contours are shown in Figure 3, along with experimental data.

Table 2: Pressure-ratio-map and efficiency-map constants for the E^3 fan.

π_{D}	a	b	k	η_{pol_o}	$\tilde{m}_o$	Δa	c	d	C	D
1.7	3.0	0.85	0.03	0.90	0.75	−0.5	3	6	2.5	15.0

The efficiency contours are shown in Figure 4.

The constants in Table 3 give a realistic high-pressure compressor map, which is compared to the E^3 compressor data. The resulting pressure-ratio contours are shown in Figure 5, along with experimental data. The efficiency contours are shown in Figure 6.

References

[1] J.L. Kerrebrock. *Aircraft Engines and Gas Turbines, 2nd Ed.* The MIT Press, Cambridge, 1996.

Table 3: Pressure-ratio-map and efficiency-map constants for the E^3 compressor.

π_D	a	b	k	η_{pol_o}	$\tilde{m}_o$	Δa	c	d	C	D
26.0	1.5	5	0.03	0.887	0.80	0.5	3	4	15.0	1.0

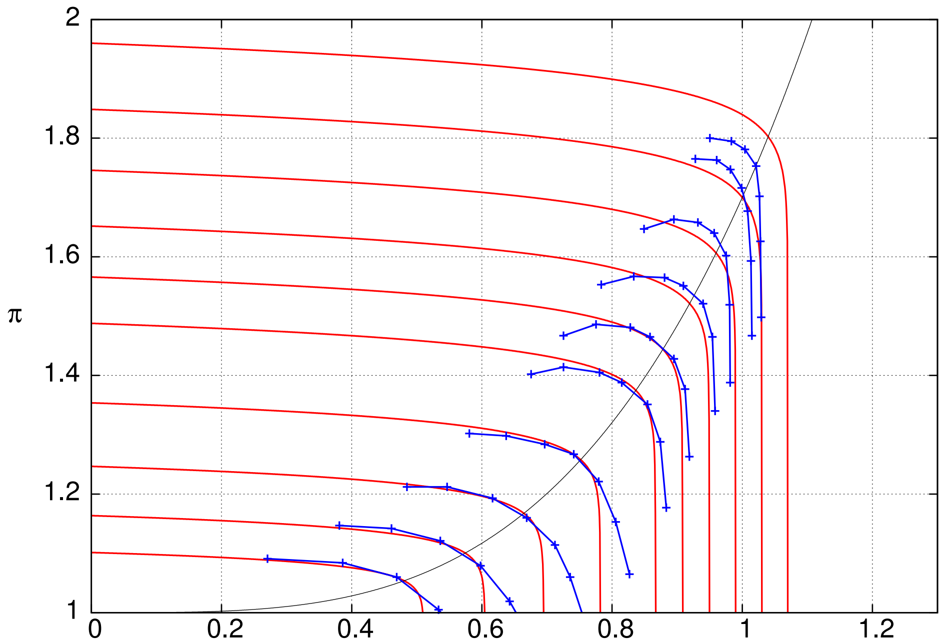

Figure 3: Pressure ratio versus normalized corrected mass flow and corrected speed, for E^3 fan. Each red line is equation (342) with E^3 fan-model constants in Table 2, and a specified experimental $\bar{N}$ value. Blue lines with symbols are measured data. Single black line is the "spine" curve $\tilde{p} = \tilde{m}^a$.

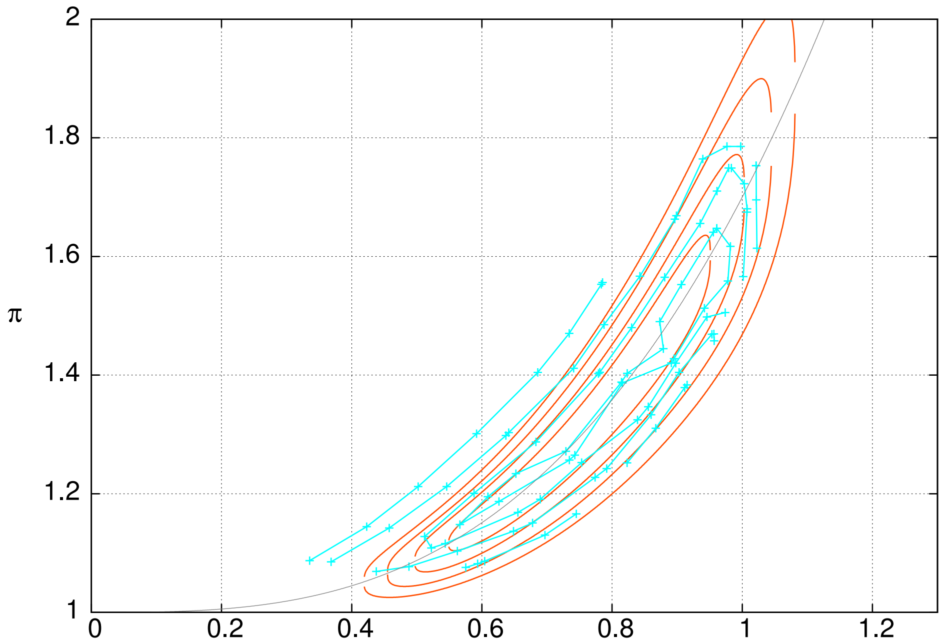

Figure 4: Polytropic efficiency contours versus corrected mass flow and pressure ratio for E^3 fan. Red lines are isocontours of equation (346) with E^3 fan-model constants in Table 2. Cyan lines with symbols are measured data. Single black line is the "spine" curve $\tilde{p} = \tilde{m}^{a+\Delta a}$.

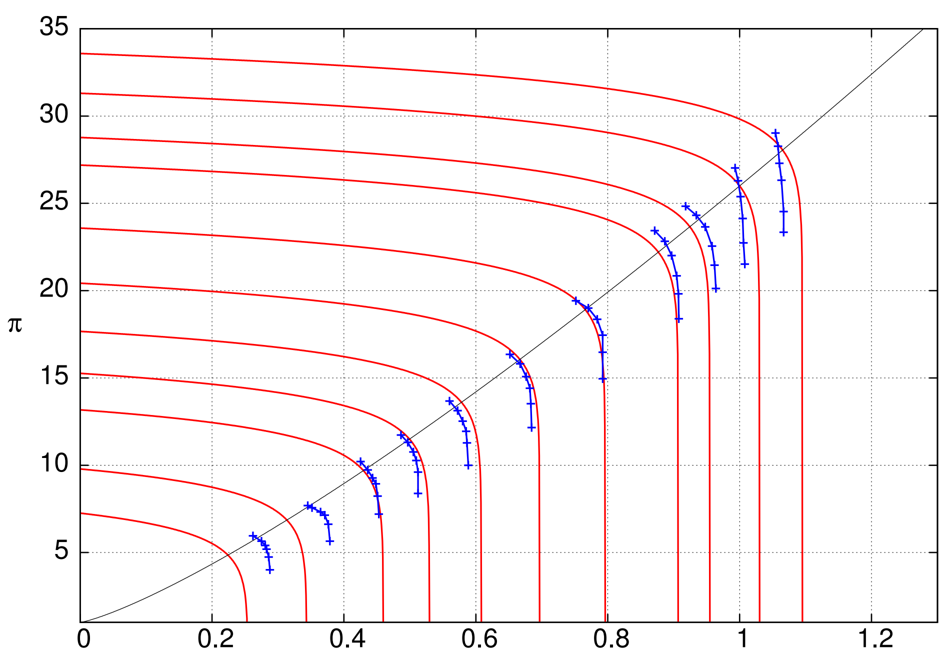

Figure 5: Pressure ratio versus normalized corrected mass flow and corrected speed for E^3 compressor. Blue lines with symbols are measured data.

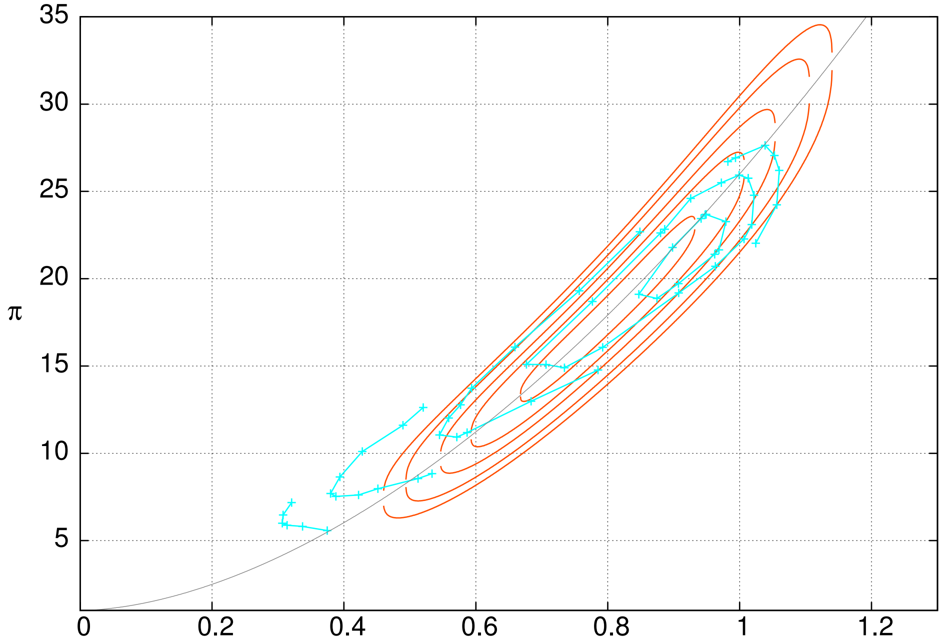

Figure 6: Polytropic efficiency contours versus corrected mass flow and pressure ratio for E^3 compressor, for compressor-model constants in Table 3. Cyan lines with symbols are measured data.