Balanced Field Length Calculation

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1 Thrust Model

A high-bypass turbofan engine, operating at fixed fan and compressor pressure ratios during the takeoff, will have its thrust decrease nearly linearly with velocity. Here, the thrust's dependence on velocity will be assumed to be quadratic, to allow an analytic derivation of the takeoff velocity versus distance.

$$F(V) \;=\; F_0 \;-\; K_V \, \tfrac{1}{2} V^2 \tag{1}$$

$$F_0 \;=\; n_{\rm eng} \, \frac{F_{\rm max} + F_{\rm ref}}{2} \tag{2}$$

$$K_V \;=\; n_{\rm eng} \, \frac{F_{\rm max} - F_{\rm ref}}{V_{\rm ref}^2} \tag{3}$$

The F_0 and K_V constants are set to give the actual thrust and also the actual thrust lapse rate at the $V_{\rm ref}$ takeoff reference airspeed, as shown in Figure 1. Because of the tangent fit at $V = V_{\rm ref}$, the exact choice of $V_{\rm ref}$ is not critical, and a suitable choice is $V_{\rm ref} = V_{\rm stall}$ for example.

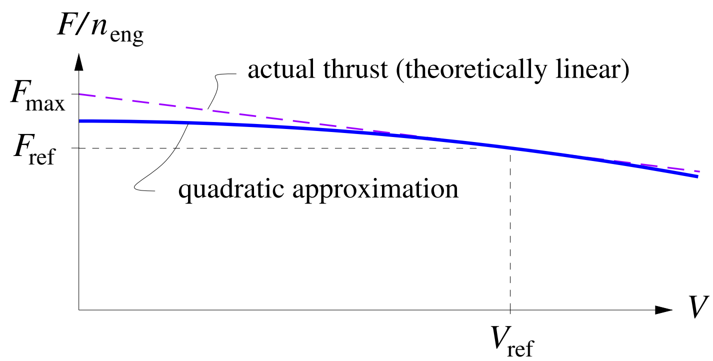

Figure 1: Thrust dependence on velocity, and assumed quadratic fit.

2 Velocity–Distance Relation

The general equation for the ground-roll acceleration or deceleration is as follows.

$$m \, \frac{{\rm d}V}{{\rm d}t} \;=\; F \;-\; mg\,\mu \;-\; \frac{1}{2}\rho V^2 \, S \, C_D \tag{4}$$

The lefthand side is manipulated to

$$\frac{{\rm d}V}{{\rm d}t} \;=\; \frac{V\,{\rm d}V}{V\,{\rm d}t} \;=\; \frac{{\rm d}(\frac{1}{2}V^2)}{{\rm d}\ell} \tag{5}$$

and the quadratic thrust model is substituted for F, converting the acceleration equation (4) to the following equivalent form.

$$\frac{\mathrm{d}(V^2)}{\mathrm{d}\ell} \;=\; 2\,\frac{F_0 - mg\,\mu}{m} \;-\; \frac{K_V + \rho S C_D}{m}\,V^2 \tag{6}$$

This is a simple 1st-order ODE for V^2, whose solution can be immediately obtained,

$$k \;=\; \frac{K_V + \rho S C_D}{m} \tag{7}$$

$$V_{\mathrm{lim}}^2 \;=\; \frac{2}{k}\,\frac{F_0 - mg\,\mu}{m} \;=\; 2\,\frac{F_0 - mg\,\mu}{K_V + \rho S C_D} \tag{8}$$

$$V^2\,(\ell) \;=\; V_{\mathrm{lim}}^2 \left[\,1 \;-\; \not{C}\,\exp(-k\ell)\,\right] \tag{9}$$

where $\not{C}$ is an appropriate integration constant.

3 Takeoff Profiles

Figure 2 shows three different takeoff profiles considered here, with the decaying-exponential V^2 plotted versus distance. Each of the three A,B,C segments is defined by (7)–(9), but with different constants. The relatively simple Normal Takeoff will be considered first. The Engine-out and Aborted Takeoffs will then be considered, as these typically set the aircraft's takeoff performance requirements.

3.1 Takeoff Length Definitions

The normal takeoff length ℓ_{TO}, shown in Figure 2, is the distance needed to accelerate to the initial climb speed V_2, which for transport aircraft is specified by FAR-25 regulation:

$$V_2 \;=\; 1.2\,V_{\mathrm{stall}} \tag{10}$$

The balanced field length ℓ_{BF} definition results from the situation where the distance needed to clear a 35 ft obstacle with one engine inoperative is equal to the distance needed to decelerate to a stop after the takeoff run is aborted. The situation is sketched in Figure 2, which also shows the as yet unknown abort-decision distance ℓ_1, and corresponding abort-decision speed V_1. For simplicity, one $V_B^2(\ell)$ function is assumed for the final roll and short climb in the Engine-out takeoff case. This is reasonable simplification, since after liftoff the lost rolling resistance is comparable to the gained induced drag, so the net thrust does not change appreciably. Table 1 gives the constant values for the three velocity segments. Reverse thrust is assumed to be unavailable for the braking segment, as required by FAR-25 regulations. $C_{D_{\mathrm{eng}}}$ is the drag of one windmilling engine, $C_{D_{i_{\mathrm{vert}}}}$ is the induced drag of the vertical tail balancing the engine-out yaw moment, and $C_{D_{\mathrm{spoiler}}}$ is the added drag of any deployed spoilers.

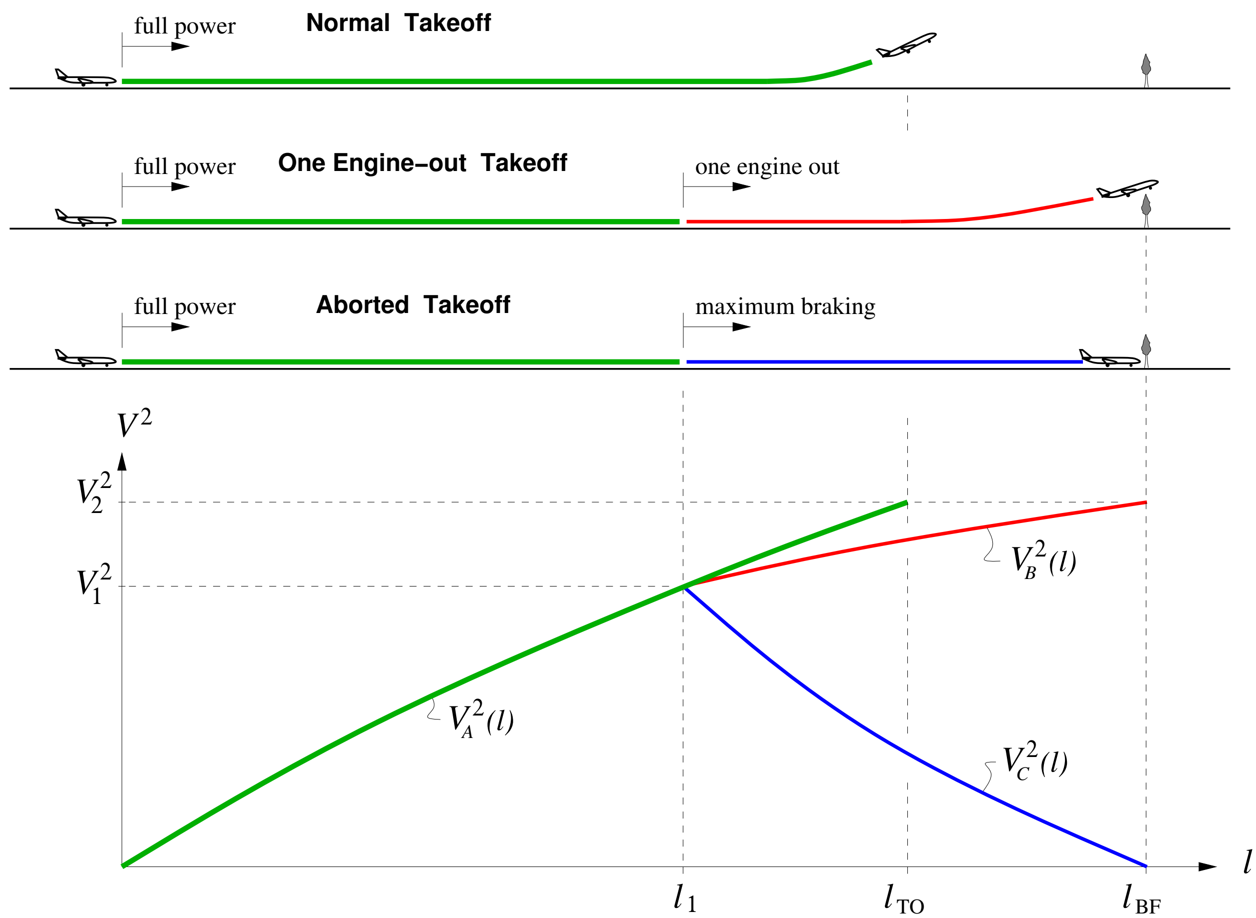

Figure 2: Engine-out takeoff and aborted takeoff over balanced-field length.

The known initial and end conditions

$$V_A(0) = 0 \tag{11}$$

$$V_B(\ell_{\rm BF}) = V_2 \tag{12}$$

$$V_C(\ell_{\rm BF}) = 0 \tag{13}$$

allow elimination of the integration constants ₵ for $V_A(\ell)$, $V_B(\ell)$, $V_C(\ell)$.

$$V_A^2(\ell) = V_{A\,\rm lim}^2 \left\{1 - \exp[-k_A \ell\,]\right\} \tag{14}$$

$$V_B^2(\ell) = V_{B\,\rm lim}^2 - \left(V_{B\,\rm lim}^2 - V_2^2\right) \exp[k_B(\ell_{\rm BF} - \ell\,)] \tag{15}$$

$$V_C^2(\ell) = V_{C\,\rm lim}^2 \left\{1 - \exp[k_C(\ell_{\rm BF} - \ell\,)]\right\} \tag{16}$$

The k_A, k_B, k_C and $V_{A\,\rm lim}^2, V_{B\,\rm lim}^2, V_{C\,\rm lim}^2$ constants are calculated via their definitions (7) and (8), using the appropriate constants in Table 1 for each of the three A,B,C cases. Note that $V_{C\,\rm lim}^2$ is negative. The balanced field length $\ell_{\rm BF}$ is still to be determined.

Table 1: Constants for computing k and $V_{\rm lim}$ for each of the three A,B,C takeoff segments.

	$V_A\,(\ell)$	$V_B\,(\ell)$	$V_C\,(\ell)$
$F_0 \;=$	$n_{\rm eng}\,\dfrac{F_{\max}+F_{\rm ref}}{2}$	$(n_{\rm eng}-1)\,\dfrac{F_{\max}+F_{\rm ref}}{2}$	0
$K_V \;=$	$n_{\rm eng}\,\dfrac{F_{\max}-F_{\rm ref}}{V_{\rm ref}^2}$	$(n_{\rm eng}-1)\,\dfrac{F_{\max}-F_{\rm ref}}{V_{\rm ref}^2}$	0
$\mu \;=$	$\mu_{\rm roll}$	$\mu_{\rm roll}$	$\mu_{\rm brake}$
$C_D \;=$	C_{D_p}	$C_{D_p}+C_{D_{\rm eng}}+C_{D_{i_{\rm vert}}}$	$C_{D_p}+n_{\rm eng}C_{D_{\rm eng}}+C_{D_{\rm spoiler}}$

3.2 Normal Takeoff Distance

The full-power Normal Takeoff distance $\ell_{\rm TO}$ is completely determined by setting $V_A(\ell_{\rm TO}) = V_2$, and solving for $\ell_{\rm TO}$.

$$\ell_{\rm TO} \;=\; -\frac{1}{k_A}\,\ln\Big[1 \,-\, (V_2/V_{A\,\rm lim})^2\Big] \tag{17}$$

The time required for the normal takeoff can also be determined by integration.

$$\begin{aligned} t_{\rm TO} \;=\; \int_0^{\ell_{\rm TO}} \frac{{\rm d}\ell}{V_A(\ell)} \;&=\; \frac{1}{V_{A\,\rm lim}} \int_0^{\ell_{\rm TO}} \frac{{\rm d}\ell}{\sqrt{1-\exp(-k_A\ell)}} \\ &=\; \frac{1}{k_A V_{A\,\rm lim}}\,\ln\left(\frac{1+\sqrt{1-\exp(-k_A\ell_{\rm TO})}}{1-\sqrt{1-\exp(-k_A\ell_{\rm TO})}}\right) \\ &=\; \frac{1}{k_A V_{A\,\rm lim}}\,\ln\left(\frac{1+V_2/V_{A\,\rm lim}}{1-V_2/V_{A\,\rm lim}}\right) \end{aligned} \tag{18}$$

The above equations require that $V_{A\,\rm lim} > V_2$, otherwise takeoff cannot occur, and $\ell_{\rm TO}$ and $t_{\rm TO}$ are not defined.

3.3 Balanced-Field Distance

The Balanced-Field case has $\ell_{\rm BF}$ and also ℓ_1 as unknowns. The necessary two equations for determining them are the A,B,C segment matching conditions at $\ell = \ell_1$,

$$V_A(\ell_1) \;=\; V_B(\ell_1) \tag{19}$$

$$V_A(\ell_1) \;=\; V_C(\ell_1) \tag{20}$$

or equivalently,

$$V_{A\,\rm lim}^2 \Big\{1-\exp[-k_A\ell_1\,]\Big\} \;=\; V_{B\,\rm lim}^2 \;-\; \Big(V_{B\,\rm lim}^2 - V_2^2\Big)\exp[k_B(\ell_{\rm BF}-\ell_1\,)] \tag{21}$$

$$V_{A\,\rm lim}^2 \Big\{1-\exp[-k_A\ell_1\,]\Big\} \;=\; V_{C\,\rm lim}^2 \Big\{1-\exp[k_C(\ell_{\rm BF}-\ell_1\,)]\Big\} \tag{22}$$

Equations (21) and (22) are now simultaneously solved for ℓ_1 and ℓ_{BF} by Newton iteration. The initial guesses $\ell_1 = \ell_{\mathrm{BF}} = \ell_{\mathrm{TO}}$ are adequate for starting the iteration. As with the normal-takeoff case, the requirement $V_{B\,\mathrm{lim}} > V_2$ is necessary for a solution to exist.

As a final step, the V_1 abort-decision speed can be calculated from any of the three segment velocities, e.g.

$$V_1^2 \;=\; V_{A\,\mathrm{lim}}^2 \left\{ 1 - \exp[-k_A \ell_1\,] \right\} \tag{23}$$